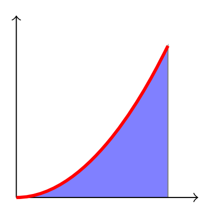
CLAREMONT McKENNA COLLEGE

Calculus of a Single Variable



Mark Huber

Fletcher Jones Foundation Associate Professor of Mathematics and Statistics and George R. Roberts Fellow

mhuber@cmc.edu

Contents

This is a course in Calculus of a single variable. The main topics of the course are:

1: Logic

2: Limits

3: Derivatives

4: Integrals

5: Series

6: Differential Equations

Wait a minute, logic? Yes! Basic mathematical definitions are all given as logical formulas, and so you will learn to read and understand logical statement. Also, you will learn the basics of several standard proof techniques. This course has the following overall structure. We will introduce problems, then formally define terms, then state and prove theorems. Please feel free to stop me at any time with questions, and often I will pause to ask questions of the class.

My lecture notes will be provided for the class, but they are my personal notes. That means that they have not been independently proofread and often contain typos. I provide them for the class for those who wish to concentrate on lecture or get ahead in the reading to make understanding easier. You can assist me and the class by letting me know when you see any mistakes in the notes, and I will correct them as quickly as possible. Class attendance is not required, but certainly recommended!

How can I get an A in this course?

1: Come to every class on time (try to come five minutes early if you are habitually late).

2: Turn off your phone/laptop/other external communication device when in lecture, they suck your attention away even when you are not looking at them.

3: Read all the homework questions and try them by yourself first (I'd schedule an hour on Wednesday afternoon or evening to do this), then talk to others or me if you have problems.

4: Do actually talk to me (or friends in the class) though if you can't do a homework problem.

5: When I hand out the study guides, learn the definitions presented.

6: Do extra practice on problem types that you find difficult. [There's a wealth of examples on the web, please ask me if you need any assistance in finding problems and we can go Google hunting together.]

1 What is Calculus?

Question of the Day What is Calculus?

Today

- What is single variable Calculus?
- What skills do you need to learn in this course?

Calculus Now, a calculus is just any system for calculating things. It has the same root in Latin as calcium, mainly because stones where used as the original calculators. Other systems of calculus include: calculus of probability, calculus of variations, multivariable calculus, propositional calculus.

In this course, we are interested in calculus of functions of a single variable. Our basic object is the function, and we will learn how to calculate properties of this function in a variety of ways.

Intuition A *single variable function* takes an *input* and returns an *output*, usually after applying one or more mathematical operations to the input. There are many different notation systems for functions.

There are multiple ways of thinking about functions, and our mathematical notation helps us with those ways.

- Argument notation: $f(x) = x^2$, $f(3) = 9$, $f(-3) = 9$
- Mapping notation: $f : x \mapsto \exp(x)$. Note $\exp(x) = e^x$, so $f(2) = e^2$.

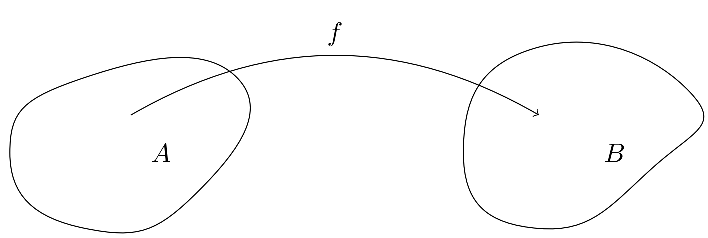
So what formally is a function? In order to give a precise definition of a function, we first start with a term that we will only define intuitively, called a set.

Set

- Intuitively, a *set* is a collection of objects, called *elements of the set*
- Sets written with curly braces
- Example: $\{\text{blue}, \text{green}, \text{red}\}$ has elements blue, green, and red. $\{1, 2, 3\}$ has elements 1, 2, and 3.
- $\in$ = "element of". So $1 \in \{1, 2, 3\}$.
- Formally, a function takes elements of one set as input, and outputs an element of a second set.

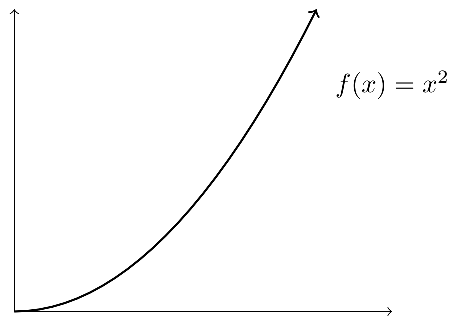
Definition 1
A **function** f from a set A to a set B (written $f : A \to B$) is a collection of ordered pairs (x, y) such that if $v = (x_1, y_1)$ and $w = (x_1, y_2)$, are two points in the set, then $y_1 = y_2$. The set of these ordered pairs is called the *graph* of the function. The first coordinate of the ordered pair is called the *input* or *argument* of the function, and the second coordinate is called the *output* or *value* of the function.

One metaphor for functions comes from archery: for each point in A, the archer stands on the point and fires exactly one arrow into B.



Example:

- Say $f : [0, \infty) \to [0, \infty)$ maps x to x^2. Then the graph is:



Models

- $v(t)$ equals velocity at time t
- $r(t)$ is interest rate at time t
- $h(a)$ is height of children age 10 given food level a
- $g(d)$ is cholesterol level given drug dosage d
- $h : C \to [0, \infty)$, C is members of this class, $h(c)$ is height of person c
- $f : \{0, 1, \ldots\} \to [0, \infty)$ is miles jogged on day t

Notation

- Blackboard boldface was invented to give a way to write a letter in bold on a blackboard. The idea was to add an extra line to the letter. So to write **Z** on a blackboard, you would write $\mathbb{Z}$ with an extra line. Over time, people realized that this was better than the bold **Z**, and so these "blackboard boldface" letters became the standard in mathematical writing. Here are a few of these that you need to know.

$$\begin{aligned}
\underbrace{\mathbb{Z}}_{\text{blackboard boldface}} &= \text{integers} \\
\mathbb{Z}^+ &= \text{positive integers} \\
\mathbb{Z}^+ \underbrace{\cup}_{\text{union}} \{0\} &= \text{nonnegative integers} \\
\mathbb{R} &= \text{real \#'s} \\
f : \mathbb{Z}^+ \cup \{0\} \to \mathbb{R} &= f \text{ is a function from nonnegative} \\
&\qquad \text{integers to the real \#'s}
\end{aligned}$$

- $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$

What is Calculus?

- Calculus allows us to answer questions about functions.
- For $f : \mathbb{R} \to \mathbb{R}$
  - Limits: What happens as the argument approaches a number?

$$\lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3} =?$$

– Derivatives: How fast is the function changing?

$$f(x) = x\exp(-x^2),\ \text{find}\ \frac{df}{dx} = f'(x),$$

– Optimization: What is the largest that $f(x)$ can be for $x \in [a, b]$? What input achieves this maximum?

$$\max_{x\in[a,b]} f(x), \quad \arg\max_{x\in[a,b]} f(x).$$

– Integrals: If the function measures a rate of accumulation, how much accumulates over time?

$$f(x) = x\exp(-x^2),\ \text{find}\ \int_a^b f(x)\ dx.$$

- Suppose $f : \mathbb{Z} \to \mathbb{R}$ given by $f(i) = 1/2^i$.
  – Series: What is the total of the function?

$$\sum_{i=1}^{\infty} f(i) = ?$$

A word about mathematics

- Several types of mathematics
- Tools for calculating things
- Intuition for when mathematics applies
- Rigorous (also called formal) proof of things
  – Euclid's elements (in geometry)
  – Logical definitions
  – Logical proof

1.1 Significant figures

Many questions have numerical answers. An answer like $\sqrt{5+\sqrt{3}}/2$ is not very intuitive when it comes to understanding how big it is. For instance, which is bigger $\sqrt{2}+\sqrt{3}$ or the constant π?

Numerical answers give better idea of what result is and so for convenience, I give all numerical, answers to four *significant digits* (4 sig figs) unless it is an integer, in which case just return the integer.

This is easiest to see by example:

number	4 sig figs
$\tau = 2\pi$	6.283
e	2.781
1.000042	1.000
$\sqrt{5+\sqrt{3}}/2$	1.297
$\sqrt{5+\sqrt{3}}/200$	0.01297

In most situations you do not have to worry about rounding, just truncate after the first four digits. The only exception is when you are using numbers as upper or lower bound. For lower bounds, always round the last digit down, for upper bounds, always round the last digit up:

$$1.297 \leq \sqrt{5+\sqrt{3}}/2 \leq 1.298.$$

2 Limits and Sequences

Question of the Day What is $\lim_{x\to 0} \sin(x)/x$?

Today

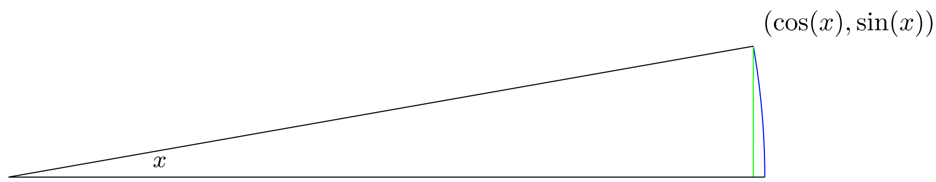
- Limits of functions
- Limits of sequences
- Infinity
- Order

2.1 Limits of functions

- General form: $\lim_{x\to a} f(x) = L$
- Intuition: When x gets close to a, $f(x)$ gets close to L
- Experimental method: try x values close to a, see what function value is.
- Example (QotD): what is $\lim_{x\to 0} \sin(x)/x$?

x	$\sin(x)/x$
0.1	0.998334
0.01	0.999983
-0.001	0.999999833333341

So $\lim_{x\to 0} \sin(x)/x = 1$. This should not be all that surprising, given that x represents the arclength in blue and $\sin(x)$ the length of the green line below.

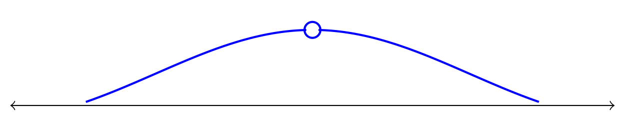


- $\sin(x)$ and x are both examples of *continuous* functions. However, $\sin(x)/x$ is not a continuous function.

Definition 2
A function f is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$.

Examples

- $\lim_{x\to 0} \sin(x) = \sin(0) = 0$.
- $\lim_{x\to 0} x = 0$.
- $\lim_{x\to 0} \frac{\sin(x)}{x} \neq \frac{0}{0}$. So the function $\sin(x)/x$ is not continuous at 0.



Intuition

- A function is continuous if you can draw its graph without lifting the pencil from the paper.
- $\sin(x)/x$ has a hole in it at 0, so is not continuous.

Fact 1 (Limits are additive and can pull out constants)
Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

- $\lim_{x\to a} f(x) + g(x) = L + M$.
- $\lim_{x\to a} cf(x) = cL$.

Example

$$\lim_{x\to 4} 3x^2 - 2\sqrt{x} = 3\lim_{x\to 4} x^2 - 2\lim_{x\to 4}\sqrt{x} = 3\cdot 4^2 - 2\sqrt{4} = 44$$

Definition 3
A function $f : A \to \mathbb{R}$ that is continuous at all $a \in A$ is called **continuous**.

Notes

- $\lim_{x\to a}$ is an example of a *mathematical operator*.
- It takes as input a function, and outputs a real number.
- A mathematical operator $\mathcal{L}$ such that

$$\mathcal{L}(c_1 f + c_2 g) = c_1\mathcal{L}(f) + c_2\mathcal{L}(g)$$

  is called a *linear operator*.
- Other linear operators in Calculus include the derivative and the integral.
- To prove limit is a linear operator, will need a formal definition of limit, which we will see in a few lectures.

Fact 2
(Limits are multiplicative and divisive) Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

- $\lim_{x\to a} f(x)g(x) = LM$
- If $M \neq 0$, then $\lim_{x\to a} f(x)/g(x) = L/M$.

Definition 4
Suppose $f : A \to B$ and $g : B \to C$. Then for $x \in A$, $[g \circ f](x) = g(f(x))$ is a function from A to C. Call $g \circ f$ the **composition** of g and f.

Fact 3
(Can compose limits) Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to L} g(x) = M$. Then $\lim_{x\to a} g(f(x)) = M$.

Fact 4
If f and g are continuous functions, then so are cf for any $c \in \mathbb{R}$, $f + g$, fg f/g (everywhere g is nonnegative) and $f \circ g$.

2.2 Limits going to infinity

- Often want to know what happens as x gets very large (say x is going to infinity)...
- ...or very small in the negative direction (say x is going to negative infinity).
- Find answer same way as with regular limits, plug in larger (or very negative) numbers to get answer
- Example: What is $\lim_{x\to\infty} 1/x$?

x	$1/x$
100	0.01
10^6	$10^{-6} = 1/1\,000\,000$

  $\lim_{x\to\infty} 1/x = 0$.

- Example: What is $\lim_{x\to\infty} \sqrt{2x}/\sqrt{x-2}$?

x	$\sqrt{2x}/\sqrt{x-2}$
100	1.42857142857
10^6	1.41421497659

  $\lim_{x\to\infty} \sqrt{2x}/\sqrt{x-2} = \sqrt{2}$

- A way to deal with ∞/∞ type limits: multiply top and bottom by inverse of leading order. Note $\sqrt{2x} = \sqrt{2}\sqrt{x}$ $\sqrt{x-2} \approx \sqrt{x}$ for large x. So multiply top and bottom by $1/\sqrt{x}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{2x}}{\sqrt{x-2}} &= \lim_{x\to\infty} \frac{(1/\sqrt{x})\sqrt{2x}}{(1/\sqrt{x})\sqrt{x-2}} \\
&= \lim_{x\to\infty} \frac{\sqrt{2}}{\sqrt{1-2/\sqrt{x}}} \\
&= \frac{\sqrt{2}}{\sqrt{1}} = \sqrt{2}.
\end{aligned}$$

- Another example of this technique: Find $\lim_{x\to\infty} \frac{x^3+3}{2x^3-1}$.
- The x^3 part dominates the top, the $2x^3$ dominates the bottom.
- Multiply top and bottom by $1/x^3$ to get:

$$\lim_{x\to\infty} \frac{(1/x^3)(x^3+3)}{(1/x^3)(2x^3-1)} = \lim_{x\to\infty} \frac{1+3/x^3}{2-1/x^3} = 1/2.$$

2.3 Limits of sequences

A sequence is just a function whose domain is $\{1, 2, 3, \ldots\}$ or $\{0, 1, 2, \ldots\}$. Only deal with limits of functions as the input goes to infinity.

- Example: $\lim_{i\to\infty} \frac{i^3+3}{2i^3-1} = 1/2$.

2.4 Order notation

- One big use of limits is in finding the *order* of a function.
- *Landau order notation* tells whether one function is eventually bigger than another, eventually smaller, or roughly the same size.
- Intuition

  - Say $f(x) = O(g(x))$ (read $f(x)$ is big-O of $g(x)$) if $g(x)$ is growing as fast or faster than $f(x)$ as x approaches 0 or infinity.
  - Say $f(x) = o(g(x))$ (read $f(x)$ is little-o of $g(x)$ if $f(x)$ is growing more slowly than $g(x)$ as x approaches 0 or infinity.
  - Say $f(x) = \Omega(g(x))$ (read $f(x)$ is big-Omega of $g(x)$ if $f(x)$ is growing as least as fast as $g(x)$ as x approaches 0 or infinity.

- How can I tell?

Fact 5
Suppose that a is either 0 or infinity, and $\lim_{x \to a} |f(x)|/|g(x)| = L$ exists. Then

- $f(x) \in O(g(x))$ as $x \to a$ iff $L \in [0, \infty)$.
- $f(x) \in o(g(x))$ as $x \to a$ iff $L = 0$.
- $f(x) \in \Omega(g(x))$ as $x \to a$ iff $L \in (0, \infty]$.

Examples

- $3n^3 + n^2 = \Theta(n^3)$ since $\lim_{n \to \infty} \frac{3n^3+n^2}{n^3} = \lim_{n \to \infty} 3 + 1/n = 3$.
- $3n^3 = o(n^4)$ since $\lim_{n \to \infty} \frac{3n^3}{n^4} = \lim_{n \to \infty} \frac{3}{n} = 0$.

Fact 6
If $f(x) \in O(g(x))$, then $g(x) \in \Omega(f(x))$.

3 Logic

Question of the Day Prove the following statement:

$$(\forall x > 2)(\exists y)(x + y > 4).$$

Today

- Logic notation
- How to prove things

True, False, other

- Some statements are true:

  All real numbers squared are at least 0.

- Some statements are false:

  All real numbers squared are at least 1.

- Some statements are neither true nor false.

  This statement is false.

- The goal of logic is to have a systematic way of showing that true and false statements are true or false.

Set notation

- A set is an unordered collection of objects
- Example: $\{\text{pencil}, \text{pen}, \text{iPad}\}$
- Example: $\{1, 2, 3\} = \{3, 2, 1\}$
- Example: $\{1, 2, 3, \ldots\}$
- Example: $[3, 10] = \{x : 3 \leq x \leq 10\}$
- Use curly braces ($\{$ and $\}$) to enclose the set
- $\in$: is an element of a set
  - blue $\in$ {green,blue,red}
  - $2.1 \in [-1, 4]$
- $A \subseteq B$ means every element of A is also in B
  - {blue,red} $\subseteq$ {green,blue,red}
  - $[2.1, 3.5] \in [-1, 4]$

Universal Quantifiers and set notation

- $\forall$: "for all" or "for all"
  - $(\forall x > 3)(2x > 6)$ is true because no matter what value of x greater than 3 I choose, $2x$ will be greater than 6.
  - $(\forall x > 3)(2x > 10)$ is false because if $x = 3.5$, then $2x = 7$ which is not greater than 10.
- $\exists$: "there exists"
  - $(\exists x > 3)(2x > 6)$ is true. For instance, if $x = 5$ then $2x = 10$. There only has to be at least one value of x that makes it true for the statement to be true.
  - $(\exists x > 3)(2x > 10)$ is true
  - $(\exists x > 3)(2x < 1)$ is false
- Can combine them:
  - $(\forall y)(\exists x)(x + y > 6)$ is true. No matter what value of y you pick, you can choose x (for instance $x = 7 - y$ works) so that $x + y > 6$.
  - $(\exists y)(\forall x)(x + y > 6)$ is false
- $:$ = "such that"
  - $(\forall x : |x| > 3)(x^2 > 9)$

How to write proofs of logic statements

- Start by *instantiating the universal quantifiers*
- Suppose I want to prove that $\forall x > 1$, something is true.
- The first line of my proof will be: Let $x > 1$.
- That *instantiates* the value of x.
- Example:

Fact: $(\forall x > 4)(x^2 + 1 > 12)$

Proof: Let $x > 4$.
Then $x^2 > 16$.
So $x^2 + 1 > 17 > 12$, and we are done. □

- The box symbol: □ indicates that the proof is complete.
- Other common ways to finish a proof are a filled in box, or QED for Quod Erat Demonstrandum (it is shown in Latin).
- Example:

Fact: $(\forall |x| \leq 3)(x + 1 < 5)$

Proof: Let $|x| \leq 3$.
Then $x \leq 3$ and $x \geq -3$.
So $x + 1 \leq 4 < 5$, and we are done. □

Proving statement with there exists

- Instantiating for exists means that you get to pick the value needed. You can choose anything that you want that works.
- Example:

Fact: $(\exists x > 2)(2x > 20)$

Proof: Let $x = 12$ (I got to pick the value of x!)
Then $2x = 24 > 20$, and we are done. $\square$

Mixing up for all and there exists

- Things get interesting once we have both $\forall$ and $\exists$
- Example: Prove that $(\forall y)(\exists x)(x + y > 6)$
  - The $\forall y$ makes the first line of the proof: Let $y \in \mathbb{R}$.
  - The $\exists x$ means we get to pick x. Since the $\exists x$ comes *after* we choose y, we get to pick x based on the value of y.
  - The goal is to make $x + y > 6$, so choose $y > 6 - x$. For instance, $y = 7 - x$ works.
  - The final proof looks like this:

Proof: Let $x \in \mathbb{R}$.
Let $y = 7 - x$.
Then $x + y = 7 > 6$, and we are done. $\square$

Negation

- Another symbol: $\neg$ turns true into false, and false into true.
- It is called the negation symbol, or logical not.
- Example: $(3 < 5)$ is true, but $\neg(3 < 5)$ is false.
- To prove a statement p is false, instead show that $\neg p$ is true.
- Negation affects universal quantifiers, by turning $\forall$ into $\exists$ and $\exists$ into $\forall$!
- For instance $\neg(\forall x > 3)$(other stuff) becomes $(\exists x > 3)\neg$(other stuff).
- In other words: to prove that a statement doesn't hold for every example, all you need is one example (called a *counterexample* where the statement doesn't hold.
- Some other examples:

Statement	Negation
$x = -2$	$x \neq -2$
$x \in [1, 2]$	$x \notin [1, 2]$
$\delta > 0$	$\delta \leq 0$
$\forall x > 2$	$\exists x > 2$
$\exists \epsilon > 0$	$\forall \epsilon > 0$

- Notice that when you negate a $\forall$ statement, you leave the rest of the expression alone: $\neg(\forall |x| > 3) = (\exists |x| > 3)$
- But when there is no universal quantifier, the expression changes: $\neg(|x| > 3) = (|x| \leq 3)$.
- To negate an entire expression, move left to right, negating as you go:

$$\neg(\exists x > 3)(|x| > 4) = (\forall x > 3)(|x| \leq 4).$$

Example:

- Prove $(\exists y)(\forall x)(x + y > 6)$ is false.
- First negate, and instead show $(\forall y)(\exists x)(x + y \leq 6)$.

Proof: Let $y \in \mathbb{R}$.
Let $x = 6 - y$.
[Note: since already set y, value of x can depend on y!]
Then $x + y = 6 \leq 6$, and we are done. $\square$

3.1 A rigorous definition

- Now that we have this notation, we can rigorously define certain concepts.
- For instance, $A \subseteq B$ means that every element of A is also an element of B. That makes the formal definition:

Definition 5
Say that A is a **subset** of B (write $A \subseteq B$) if

$$(\forall a \in A)(a \in B).$$

4 Rigorous Limits

Question of the Day Prove that $\lim_{x\to 3} 2x = 6$.

Today

- The logical definition of limit.
- Using the definition to prove true/false statements about limits.
- Increasing and decreasing functions.

Defining limits

- The limit as x approaches a of $f(x)$ is L.

$$\lim_{x\to a} f(x) = L.$$

- Intuition: When x is close to a, then $f(x)$ is close to L.
- Use ϵ to measure how close $f(x)$ is to L.

$$|f(x) - L| \leq \epsilon$$

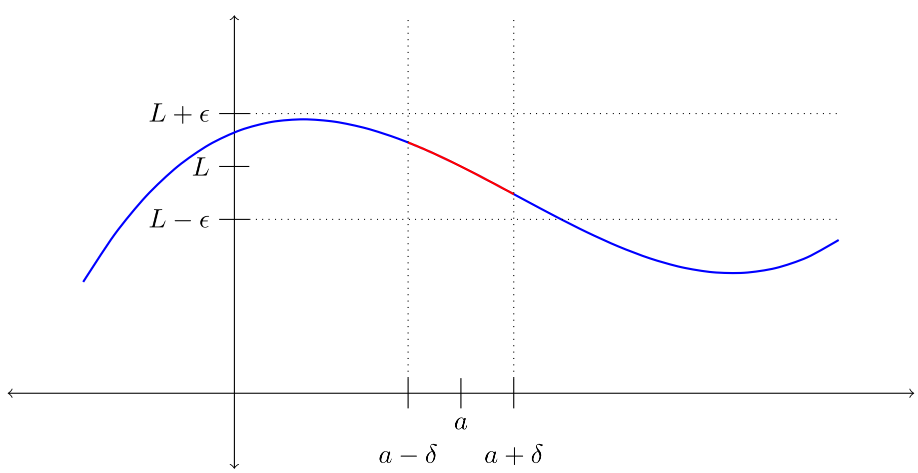
- Use δ to measure how close x is to a.

$$|x - a| \leq \delta$$

Definition 6
For $f : A \to B$, the **limit** of $f(x)$ as x approaches a is L (written $\lim_{x\to a} f(x) = L$) means

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in A : |x - a| \leq \delta)(|f(x) - L| \leq \epsilon).$$

- Here is the picture.



The red part of the function lies between the horizontal lines $L - \epsilon$ and $L + \epsilon$.

Proving facts about limits

- "For all" needs instantiating
- $\forall \epsilon > 0$ becomes "Let $\epsilon > 0$"
- $\exists \delta > 0$ becoems "Let $\delta =$" and you can choose what goes on the right hand side at a later time.

Example: Show that $\lim_{x\to 3} 2x = 6$

Proof: Let $\epsilon > 0$
Let $\delta =$ [leave blank for now]
Let x satisfy $|x-3| < \delta$
Then $-\delta < x-3 < \delta$
[Goal: $|2x-6| < \epsilon$ which means $-\epsilon < 2x-6 < \epsilon$]
So $-2\delta < 2x-6 < 2\delta$
[At this point, make $2\delta = \epsilon$ by making $\delta = \epsilon/2$]
Since $\delta = \epsilon/2$, $-\epsilon < 2x-6 < \epsilon$, and we are done! □

Final proof:

Proof: Let $\epsilon > 0$
Let $\delta = \epsilon/2$
Let x satisfy $|x-3| < \delta$
Then $-\delta < x-3 < \delta$
[Goal: $|2x-6| < \epsilon$ which means $-\epsilon < 2x-6 < \epsilon$]
So $-2\delta < 2x-6 < 2\delta$
Therefore $-\epsilon < 2x-6 < \epsilon$, and we are done! □

Example: a harder example

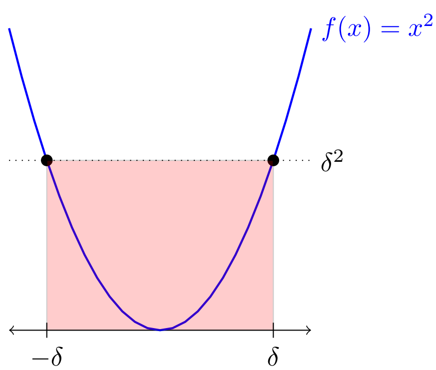
- Show that $\lim_{x\to 2} x^2 = 4$.
- First four lines of proof don't require thought:

  Proof: Let $\epsilon > 0$
  Let $\delta =$ [save choice for later]
  Let x satisfy $|x-2| < \delta$
  Then $-\delta < x-2 < \delta$
- And the last line is the goal that is determined for us:

  Proof: Therefore $-\epsilon < x^2-4 < \epsilon$, and we are done!
- So the question is how to get from $-\delta < x-2 < \delta$ over to $-\epsilon < x^2-4 < \epsilon$. Sometimes easier to do in two pieces, given $x-2 < \delta$ and $-\delta < x-2$, show that $x^2-4 < \epsilon$ and $-\epsilon < x^2-4$.
- Would like to square $-\delta < x-2 < \delta$.
- Note: Can't just square all three things!

$$(-\delta)^2 < (x-2)^2 < \delta^2 \Leftrightarrow \delta^2 < (x-2)^2 < \delta^2$$

  is a false statement.
- Consider the picture:



$$\begin{aligned}
-\delta < x-2 < \delta &\Rightarrow 0 \le (x-2)^2 < \delta^2 \\
&\Rightarrow 0 \le x^2 - 4x + 4 < \delta^2 \\
&\Rightarrow 4x - 8 \le x^2 - 4 < \delta^2 + 4x - 8 \\
&\Rightarrow 4(x-2) \le x^2 - 4 < \delta^2 + 4(x-2) \\
&\Rightarrow 4(-\delta) < x^2 - 4 < \delta^2 + 4\delta
\end{aligned}$$

- Need $-\epsilon \le -4\delta$ and $\delta^2 + 4\delta \le \epsilon$
- Making $\delta \le \epsilon/4$ gives $-\epsilon \le -4\delta$.
- Making $\delta \le 1$ makes $\delta^2 \le \delta$ and $\delta^2 + 4\delta \le 5\delta$.
- Making $\delta \le \epsilon/5$ gives $5\delta \le \epsilon$.
- To make all this work,
$$\delta = \min\{1, \epsilon/4, \epsilon/5\} = \min\{1, \epsilon/5\}$$
- Now that I know what δ to use, we can write the proof out moving forward.

  Proof: Let $\epsilon > 0$.
  Let $\delta = \min\{1, \epsilon/5\}$.
  Let x satisfy $|x-2| < \delta$.
  Then $-\delta < x - 2 < \delta$.
  And $0 \le (x-2)^2 < \delta^2$.
  So $0 \le x^2 - 4x + 4 < \delta^2$.
  Adding $4(x-2)$ to everything gives
  $4(x-2) \le x^2 - 4 < \delta^2 + 4(x-2)$.
  Using $-\delta < x - 2 < \delta$ then gives $4(-\delta) < x^2 - 4 < \delta^2 + 4\delta$.
  Since $\delta \le \epsilon/5$, $-\epsilon < -4\delta$.
  Since $\delta \le 1$, $\delta^2 + 4\delta \le 5\delta$.
  Since $\delta \le \epsilon/5$, $5\delta \le \epsilon$.
  So $\delta^2 + 4\delta \le \epsilon$.
  Which means $-\epsilon < x^2 - 4 < \epsilon$, and we are done. $\square$

Comments

- Called the ϵ, δ definition of limits.
- Only need to do formal proof of limits if asked.
- Otherwise, just invoke the continuity of $f(x) = x^2$ to get $\lim_{x\to 2} x^2 = 2^2 = 4$.
- Not all limits exist! (Need new definition if limit is infinity.)

$$\lim_{x\to 0} 1/x = ?, \quad \lim_{x\to\infty} x = ?$$

5 Derivative

Question of the Day What is the derivative of $f(x) = x^2$ at $x = 1$?

Today

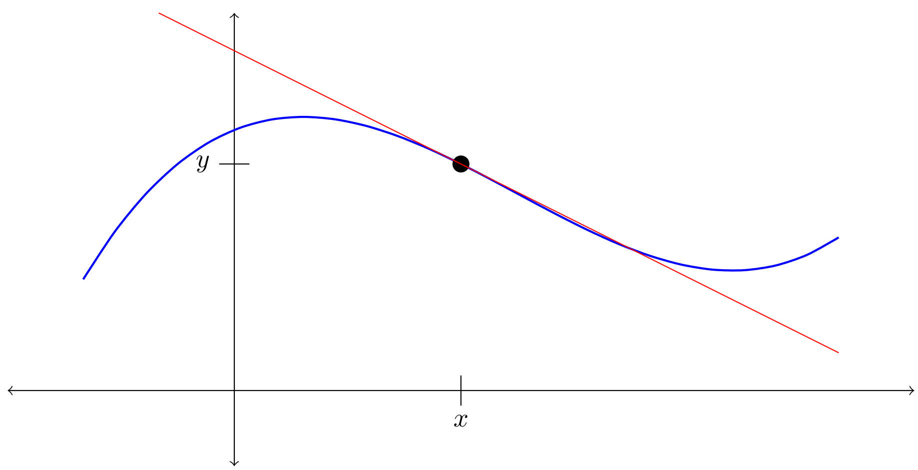
- The notion of a differential.
- The intuitive notion of a derivative.
- The derivative as a linear approximation to a function.
- The Product Rule

Notation for the derivative Say $y = f(x)$. Then the derivative of f is

$$f'(x) = y' = [f]' = \frac{dy}{dx} = \frac{df}{dx}.$$

Notions of the derivative

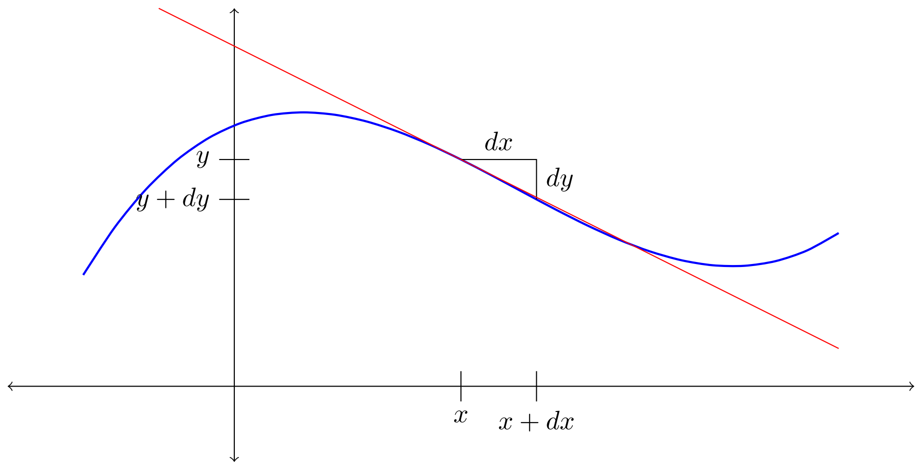
- There are several ways to think about the derivative.
- Geometric: slope of a curve



- Marginal change: how does small change in x change $f(x)$?
- Instananeous rate of change: derivative of position is velocity, derivative of velocity is acceleration
- Ratio of infinitesimal quantities
  - dx represents the infinitesimally small change in x
  - dy represents the infinitesimally small change in y
  - Regular slope of a line is rise/run (change in y/change in x)
  - Derivative is dy/dx
  - Derivative gives best approximation to a function with a line:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \underbrace{o(x - x_0)}_{\text{error}}$$

  (Gives a derivative in higher dimensions.)



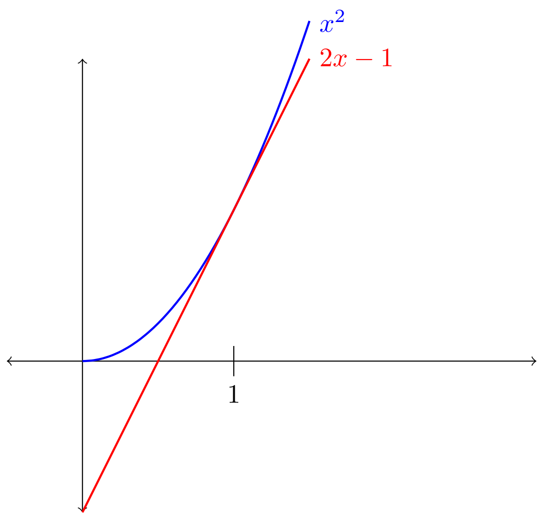
Definition 7
A function f is a **line** if it has the form $f(x) = mx + b$ for constants m and b. Call m the *slope* of the line and b the y-intercept.

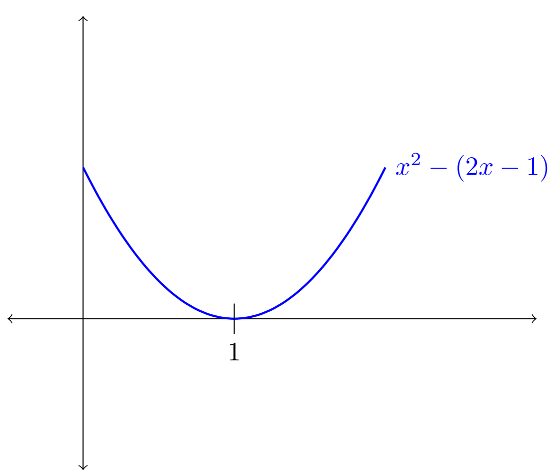
Derivative of x^2

- Claim that $[x^2]' = 2x$.
- So answer to QotD is $2(1) = 2$.
- So best approximation to x^2 using $x_0 = 1$ is

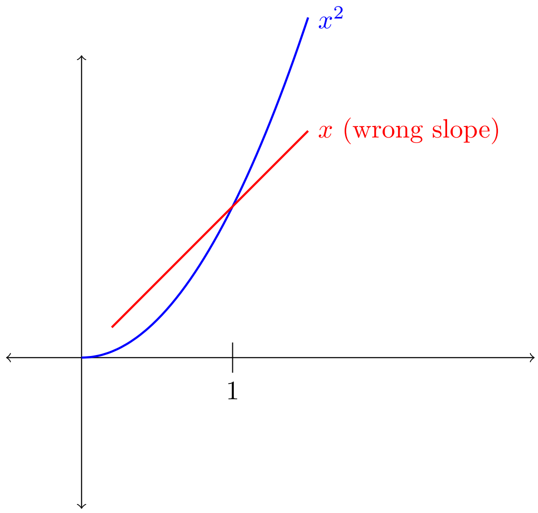
$$f(1) + (x - 1)f'(1) = 1 + (x - 1)2 = 2x - 1$$

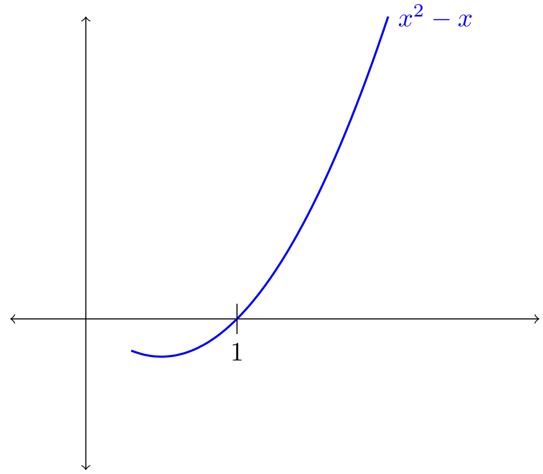
- Visually:





Difference flat at 0





Difference not flat at 0

5.1 Properties of the derivative

Derivative of a line is just the slope

Fact 7
The derivative of a line is just the slope of the line

$$[mx + b]' = m.$$

Example: $[3x - 2]' = 3$, $[x + 4]' = 1$, $[5]' = 0$.

Lemma 1
The derivative is a linear operator. So for any two differentiable functions f and g and real constants a and b,

$$[af + bg]' = a[f]' + b[g]'.$$

Intuition

- The function af is just a scaling of the vertical coordinate by a. So the slope should scale by the same amount.

- Suppose

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)$$
$$g(x) = g(x_0) + (x - x_0)g'(x_0) + o(x - x_0).$$

  Then

$$f(x) + g(x) = f(x_0) + g(x_0) + (x - x_0)(f'(x_0) + g'(x_0)) + o(x - x_0).$$

  So

$$[f + g](x) = [f + g](x_0) + (x - x_0)[f' + g'](x_0) + o(x - x_0).$$

  Hence

$$[f + g]' = f' + g'.$$

Lemma 2 (Product Rule)
If f and g are differentiable functions, then so is $f \cdot g$, and

$$[fg]' = fg' + f'g.$$

Fact 8
The derivative of x^2 is $2x$.

Proof. From our previous facts and lemmas:

$$\begin{aligned}
[x^2]' &= [x \cdot x]' \\
&= x[x]' + [x]'x \\
&= x(1) + (1)x \\
&= 2x.
\end{aligned}$$

□

Fact 9
The derivative of x^3 is $3x^2$.

Proof. From the product rule and the derivative of x^2:

$$[x^3]' = [x \cdot x^2]' = x[x^2]' + [x]'x^2 = x(2x) + (1)x^2 = 3x^2.$$

□

Lemma 3 (General Product Rule)
Let $f_1, \ldots, f_k$ be differentiable functions. Then so is $f_1 \cdots f_k = \prod_{i=1}^k f_i$, and

$$[f_1 \cdots f_k]' = [f_1]' \prod_{i \neq 1}^{k} f_i + [f_2]' \prod_{i \neq 2}^{k} f_i + \cdots + [f_k]' \prod_{i \neq k} f_i.$$

Fact 10
If $n \in \{1, 2, \ldots\}$, then $[x^n]' = nx^{n-1}$.

Proof is by induction.

- There are two parts to an induction proof.
- The base case: prove that the statement holds for the smallest value of n
- The induction step. Assume that the statement is true for n. Then if it holds for $n+1$, then the statement is true for all n.

Proof. Proof by induction. When $n = 1$, $nx^{n-1} = 1 \cdot x^{1-1} = 1$, which is $[x]'$, so the base case holds.
Assume the statement is true for n. Then for $n + 1$:

$$[x^{n+1}]' = [x \cdot x^n]' = x[x^n]' + [x]'x^n = x(nx^{n-1}) + x^n = (n+1)x^n.$$

This completes the induction. □

6 Rigorous Derivatives

Question of the Day What is the rigorous definition of a derivative

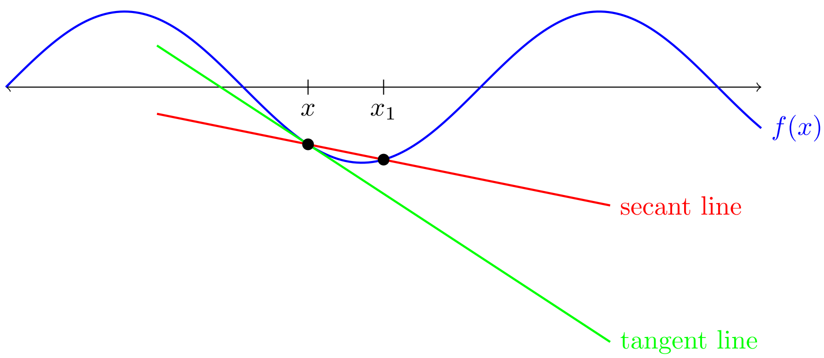
Today

- The formal definition of a derivative.
- Using the formal definition to prove facts about the derivative.
- Six derivative rules to memorize.

What is the derivative?

Definition 8
A line that passes through points $(x, f(x))$ and $(x_1, f(x_1))$ is called a *secant line*

[Secant comes from the Latin verb Secare "to cut"]
As x_1 gets closer to x, get closer to the *tangent* line at x.



Definition 9
Say that $g(x_1)$ is the **tangent line** to f at x if

$$f(x_1) - g(x_1) = o(x - x_1).$$

In this case, $o(x - x_1)$ means that

$$\lim_{x - x_1 \to 0} \frac{f(x_1) - g(x_1)}{x - x_1} = 0.$$

Finding the tangent line

- So how do we find this tangent line?
- The tangent line is the best linear approximation to the function.
- Slope of tangent line
- For historical reasons, let $x_1 - x = h$. Then as $x_1 \to x$, $h \to 0$. And $x_1 = h + x$.

Definition 10
The **derivative of f at x** is

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

when this limit exists.

Note: the derivative of a function at an x value is a number.

Definition 11
If $f' = g$ for all $x \in (a, b)$, say that g is the **derivative of f**.

Note: the derivative of a function is another function.

Derivatives are linear operators

Definition 12
Call $\mathcal{L}$ a **linear operator** if for scalars a and b and vectors v and w:

$$\mathcal{L}[av + bw] = a\mathcal{L}[v] + b\mathcal{L}[w].$$

- Let c_1 and c_2 be constants
- Let f and g be functions
- Limits are real operators means that if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and are finite, then

$$\lim_{x \to a} [c_1 f + c_2 g](x) = c_1 \lim_{x \to a} f(x) + c_2 \lim_{x \to a} g(x)$$

- Deriviatives are linear operators means that if f' and g' exist, then

$$[c_1 f + c_2 g]' = c_1 f' + c_2 g'.$$

- Now that we have rigorous definitions of limit and derivatives, can prove this!

Fact 11 (Limits scale)
Suppose that $\lim_{x \to a} f(x) = L$ and $c \in \mathbb{R}$. Then $\lim_{x \to a} c \cdot f(x) = cL$.

Proof: Case I: Suppose $c = 0$.
Then $cf(x) = 0$ for all x, and the limit is $0 = 0 \cdot L$
since it is a constant function.
Case II: Suppose $c > 0$
Let $\epsilon > 0$.
Then let $\epsilon' = \epsilon/c$. Note $\epsilon' > 0$.
Then by the definition of limit there exists δ such that
$|x - a| \leq \delta \to |f(x) - L| < \epsilon'$
Let x be any number with $|x - a| \leq \delta$.
Then $-\epsilon/c < f(x) - L < \epsilon/c$
Multiplying by c gives $-\epsilon < cf(x) - cL < \epsilon$
Hence $\lim_{x \to a} cf(x) = L$
Case III: Suppose $c < 0$
Let $\epsilon > 0$
Now $\epsilon' = -\epsilon/c > 0$, and the proof proceeds as in Case I. $\square$

Structure of proof

- Started with $\epsilon > 0$ to prove $\lim_{x \to a} cf(x) = cL$.
- Used fact that $\lim_{x \to a} f(x) = L$ to get our δ value!
- Common way to prove these types of limit results.

Fact 12
(Limits add) Supose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then $\lim_{x\to a}[f+g](x) = L+M$.

Proof. Let $\epsilon > 0$. Then there exists δ_f such that

$$|x-a| < \delta_f \rightarrow |f(x) - L| \leq \epsilon/2.$$

Also, there exists δ_g such that

$$|x-a| < \delta_g \rightarrow |g(x) - M| \leq \epsilon/2.$$

Let $\delta = \min\{\delta_f, \delta_g\}$. Let x be such that $|x-a| < \delta$. Then $|f(x) - L| < \epsilon/2$ and $|g(x) - M| < \epsilon/2$. Recall that for any $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$ (called the triangle inequality for absolute value.) So

$$\begin{aligned}
|[f+g](x) - (L+M)| &= |f(x) - L + (g(x) - M)| \\
&\leq |f(x) - L| + |g(x) - M| \\
&\leq \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

$\square$

- Combining these two facts shows that limits are a linear operator!
- One you have it for limits, for derivatives it is easy

Proof that derivatives are linear operators. Let x be any real number. Then

$$\begin{aligned}
af'(x) + bg'(x) &= a \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + b \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \left(a\frac{f(x+h) - f(x)}{h} + b\frac{g(x+h) - g(x)}{h} \right) \\
&= \lim_{h\to 0} \frac{[af+bg](x+h) - [af+bg](x)}{h},
\end{aligned}$$

so the derivative of $[af+bg]$ exists at x, and must equal $af'(x) + bg'(x)$. $\square$

7 More ways of finding derivatives

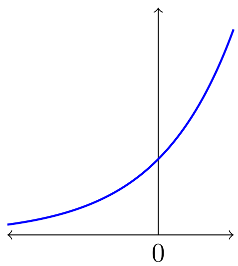
Question of the Day What is the derivative of $(x^2+1)^{-2}$?

Today

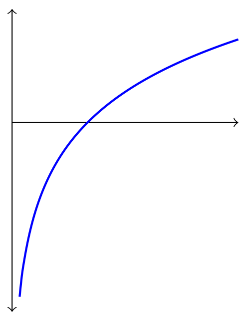
- Six derivative rules to memorize.
- The chain rule

Functions you should know story behind

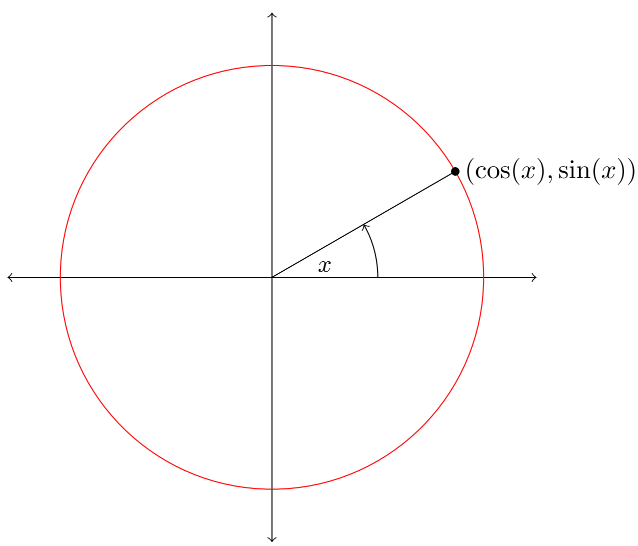
- Exponential function: $\exp(x)$ is something like e^x.



- Natural logarithm (natural log): $\ln(x)$ is the inverse of the exponential function. $[\exp(\ln(x)) = \ln(\exp(x)) = x]$

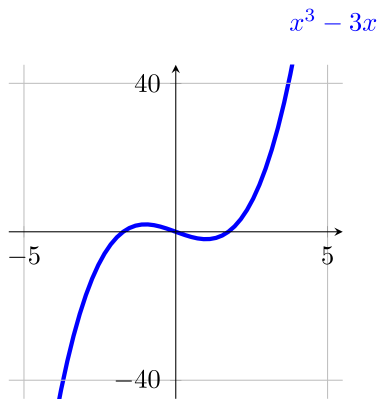


- Sine, Cosine, Tangent: $(\cos(x), \sin(x))$ coordinates of point on the unit circle angle x counter-clockwise from right hand horizontal axis. Tangent is slope of line from origin to this point.



$\tan(x) = \frac{\sin(x)}{\cos(x)}$

- Arcsine, arccosine, arctangent: Inverse functions of trig functions are named arc[function] or written $\text{function}^{-1}(x)$.
- Polynomials: linear combinations of x^i for $i \in \{0, 1, 2, \ldots\}$



- Formal definition of each of these functions very different from intuition!

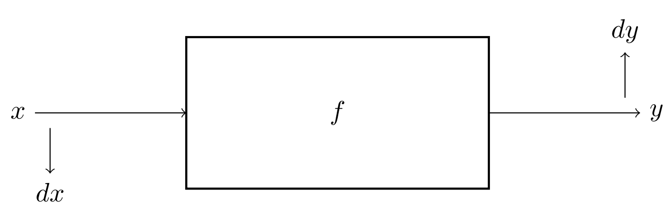
7.1 Six derivatives to memorize

- Helpful to have these derivatives memorized.
- They come up the most often in using Calculus.

$f(x)$	$f'(x)$
x^a	ax^{a-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\exp(x)$	$\exp(x)$
$\arctan(x)$	$1/(1+x^2)$
$\ln(x)$	$1/x$

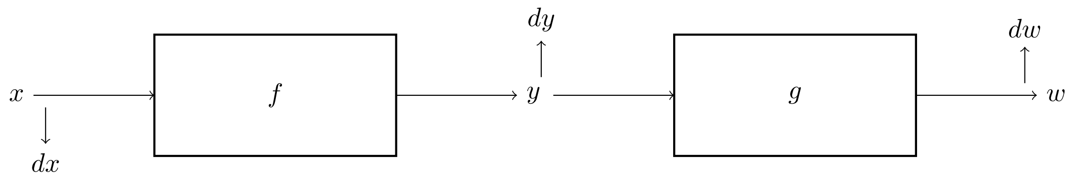
Example

- What does $[\ln(x)]' = 1/x$ mean?
- Well, $\ln(2) \approx 0.69314718056\ldots$...
- Change 2 to 2.01 is changing the x coordinate by 0.01...
- So
$$\ln(2.01) \approx \ln(2) + 0.01[\ln]'(2) = \ln(2) + 0.01(1/2) = 0.698147\ldots$$
- Only an approximation, $\ln(2.01) = 0.698134\ldots$, but accurate to first four decimal places!
- The picture: change x a little bit dx, and y changes a little bit dy, $dy = dx f'(x)$.



7.2 The Chain Rule

- The way the chain rule works is suppose y is fed into another function g to get w.
- Question: how much does w change when x changes?



- So x changes y changes w.
- Now $dw = dy \cdot g'(y) = dy \cdot g'(f(x))$ and $dy = dx \cdot f'(x)$. So

$$dw = dx \cdot g'(f(x))f'(x) \Rightarrow \frac{dw}{dx} = g'(f(x))f'(x).$$

Lemma 4 (The Chain Rule)
There are several ways to state the chain rule. Using composition notation:

$$[g \circ f]' = [g' \circ f] \cdot f'$$

Using argument notation:

$$[g(f(x))]' = g'(f(x))f'(x).$$

Using differential notation

$$\frac{dw}{dx} = \frac{dw}{dy} \cdot \frac{dy}{dx}.$$

Example

- Find the derivative of $\exp(3x)$.
- Formal approach: set f and g, then use formula

$$f(x) = 3x, g(y) = \exp(y) \Rightarrow f'(x) = 3, g'(y) = \exp(y)$$

- So

$$g'(f(x))f'(x) = \exp(3x) \cdot (3) = 3\exp(3x).$$

QotD

- What is the derivative of $(x^2+1)^{-1}$?
- Set up f and g, then use the formula

$$f(x) = x^2 + 1, g(y) = y^{-1} \Rightarrow f'(x) = 2x, g'(y) = -1y^{-1-1} = -y^{-2}$$

-
$$[g(f(x))]' = g'(f(x))f'(x) = -(x^2+1)^{-2}(2x) = -\frac{2x}{(x^2+1)^2}.$$

Note, derivative of $\exp(x)$, $\ln(x)$, and chain rule give the power rule.

Lemma 5
For all $a \in \mathbb{R}$ and $x \neq 0$, the derivative of x^a is ax^{a-1}.

Proof. Suppose $x \neq 0$. Since exp and ln are inverse functions, $x = e^{\ln(x)}$. Hence $x^a = (e^{\ln(x)})^a = e^{a\ln(x)}$ by the rules of exponents. This is a compositon of functions!

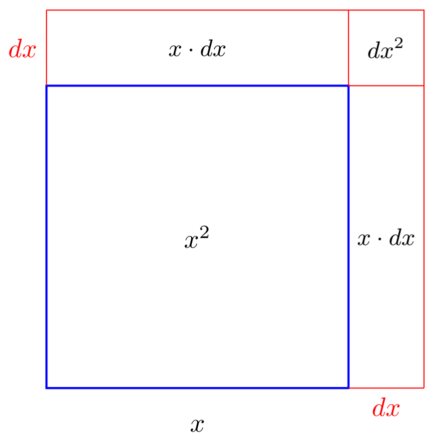
$$f(x) = a\ln(x), g(y) = e^y \Rightarrow f'(x) = a/x, g'(x) = e^y.$$

So

$$[x^a]' = [e^{a\ln(x)}]' = e^{a\ln(y)}(a/x) = x^a(a/x) = ax^{a-1}.$$

$\square$

Geometric picture of power rule for $a = 2$:



$$d(x^2) = x \cdot dx + x \cdot dx + (dx)^2 = 2x \cdot dx$$

$$\frac{d(x^2)}{dx} = 2x$$

8 Setting up Integrals

Question of the Day Water is flowing into a pool at rate $20t$ gallons per minute. How much water comes in the first hour?

Today

- Applications of integration

What is mathematics

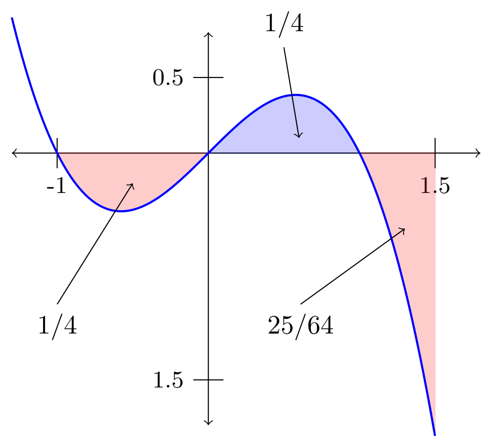
- Mathematics is abstraction: taking two dissimilar problems, and seeing the commonality in each
- If my company owns 5 airplanes and I buy 3 more, how many airplanes do I have? $5 + 3 = 8$
- If I own 5 cows and I buy 3 more, how many cows do I have? $5 + 3 = 8$
- Cows are not airplanes, but that doesn't matter to the problem
- Integrals (like numbers and addition) apply to a wide variety of problems that at first do not look alike

What are integrals?

- The integral of $f(x)$ over $[a, b]$. Write

$$\int_{x\in[a,b]} f(x)\ dx = \int_{x=a}^{b} f(x)\ dx = \int_{a}^{b} f(x)\ dx.$$

- Geometric interpretation: The area above the x-axis minus the area below the x-axis



Integral $= -(1/4) + (1/4) - (25/64)$

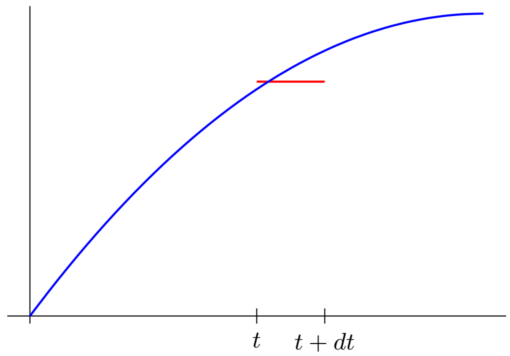
$= -25/64 \approx -0.3906$

- Accumulation: when stuff (water, oil, distance) is accumulating at a rate $f(t)$ over time $t \in [a, b]$.

Accumulation

- A car is traveling 60 miles/hour for 1/2 hour. How far does it travel?
- Water is flowing at 60 gallons/minute for 1/2 minute. How much water accumulates?
- In general: accumultation = rate times time
- Works when the rate $r(t)$ does not vary with time
- In qotd, rate is certainly varying with time

- Idea: only work over a small time interval $[t, t + dt]$



- Over this small interval, rate is effectively constant
- Accumulation over interval from t to $t + dt$ is

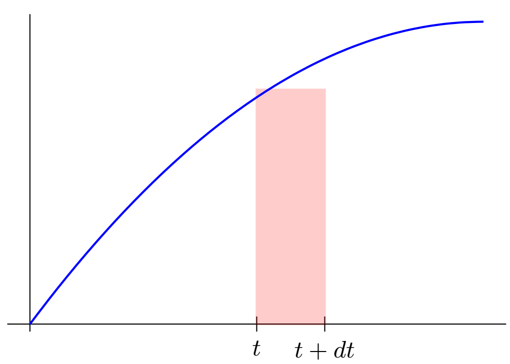
$$r(t) \cdot (t + dt - t) = r(t)\ dt.$$

- Integrate comes from Latin to "make whole"
- To get the whole accumulation, integrate over time values
  (integral sign is a stretched out S for summation)

$$\int_{t=a}^{b} r(t)\ dt.$$

Connection to geometry

- Note that $r(t)\ dt$ also approximates the area under the curve from t to $t + dt$ (or negative the area if $r(t) < 0$.)



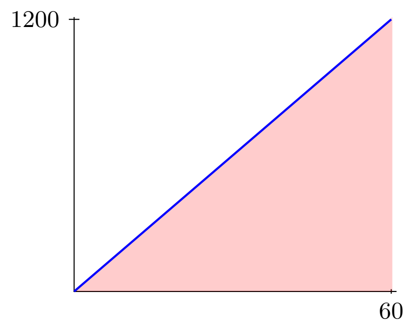
- For infinitesimal dt, and $r(t) \geq 0$, get area under the curve

Use connection to solve qotd

- From our discussion on accumulation, answer to qotd is

$$\int_{t=0}^{60} 20t\ dt.$$

- From geometry, area of shaded region:



- This is a triangle with height 1200, base 60, so total area

$$1200 \cdot 60 \cdot (1/2) = 36\,000 \text{ gallons}$$

Properties of integrals What properties should integrals obey?

- Respect inequalities. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_{x\in[a,b]} f(x)\ dx \leq \int_{x\in[a,b]} g(x)\ dx.$$

- Linear operators. If $\int_{x\in[a,b]} f(x)\ dx$ and $\int_{x\in[a,b]} g(x)\,\mathrm{d}\,x$ exist, and c_1 and c_2 are two real numbers, then

$$\int_{x\in[a,b]} c_1 f(x) + c_2 g(x)\ dx = c_1 \int_{x\in[a,b]} f(x)\ dx + c_2 \int_{x\in[a,b]} g(x)\ dx.$$

- Accumulation: if water is flowing at slower rate f than faster rate g, then total water from f rate is less than total water from g rate.
- Geometry: If $f \leq g$, then positive area under f is less than positive area under g, and negative area under f greater than negative area under g, so positive area - negative area under f is less than positive area - negative area under g.

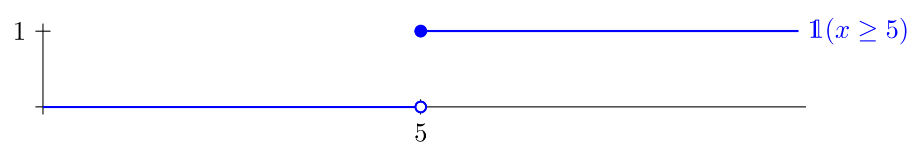
8.1 Indicator and simple functions

- Useful for describing functions that are constant over intervals

Definition 13
The indicator function $\mathbb{1}(\cdot)$ takes an argument that is either true or false. If the argument is true it returns a 1, otherwise it returns a 0.

Example

- Let $f(x) = \mathbb{1}(x \geq 5)$
- Then $f(6) = \mathbb{1}(6 \geq 5) = 1$ because $6 \geq 5$ is true
- But $f(4) = \mathbb{1}(4 \geq 5) = 0$ because $4 \geq 5$ is false
- Graph of the function looks like this:



- Filled circle at $(5, 1)$, and empty circe at $(5, 0)$ since $\mathbb{1}(5 \geq 5) = 1$.

Using indicators

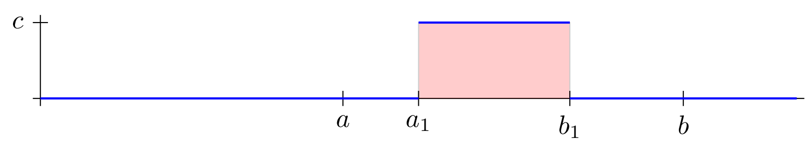
- Suppose I want a function that is 5 or 0, not 1 or 0

$$5 \cdot \mathbb{1}(x \in [a, b])$$

- Indicator functions are easy to integrate!
- First integral definition:

Definition 14
Suppose $a \le a_1 \le b_1 \le b$ then

$$\int_{x\in[a,b]} c\mathbb{1}(x \in [a_1, b_1])\ dx = c(b_1 - a_1)$$



Definition 15
For $a \le b$, suppose $f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$, where $f_i(x) = \mathbb{1}(x \in [a_i, b_i))$, and $(\forall i)(c_i \in \mathbb{R}$ and $a \le a_i \le b_i \le b)$. Then f is a **simple** function.

Our second integral definition:

Definition 16
Let $f(x) = f_1(x) + \cdots + f_n(x)$ be a simple function. Then

$$\int_{x\in[a,b]} f(x)\ dx = \int_{x\in[a,b]} f_1(x)\ dx + \cdots + \int_{x\in[a,b]} f_n(x)\ dx.$$

Example

- What is $\int_{x\in[0,3]} 2\mathbb{1}(x \in [0, 1]) - \mathbb{1}(x \in [1, 3])\ dx$?

$$\int_{x\in[0,3]} 2 \cdot \mathbb{1}(x \in [0, 1])\ dx + \int_{x\in[0,3]} (-1)\mathbb{1}(x \in [1, 3])\ dx =$$
$$2(1 - 0) + (-1)(3 - 1) = 0.$$

9 Rigorous Integrals

Question of the Day What is the definition of $\int_a^b f(x)\ dx$ for general functions f?

Today

- Rigorous definition of the Riemann integral
- A function with no Riemann integral

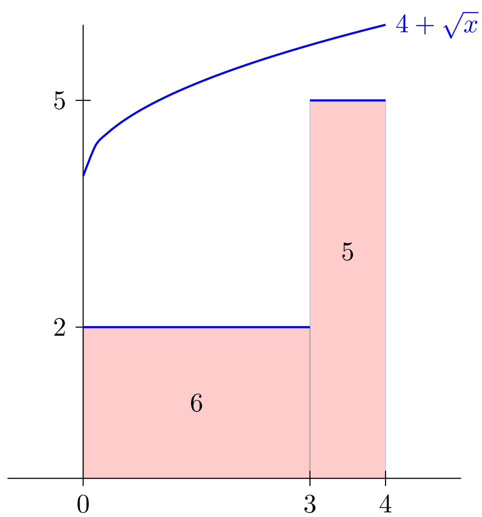
Integration is an idea

- Area above x-axis minus area below x-axis
- Accumulation at a rate over a time interval
- More than one way to define integral
- Some more general than others
- Today: Riemann integral
- More advanced, Lebesgue, Steljies, Ito

Idea

- Remember the integral is defined for simple functions
- Example:

$$\int_0^4 2 \cdot \mathbb{1}(x \in [0,3)) + 5 \cdot \mathbb{1}(x \in [3,4])\ dx = 2(3-0) + 5(4-3) = 11$$



- Easy to show facts about integral of simple functions.

Lemma 6
Let s_1 and s_2 be two simple functions over $[a,b]$, and $\alpha, \beta \in \mathbb{R}$. Then

$$\int_a^b \alpha s_1(x) + \beta s_2(x)\ dx = \alpha \int_a^b s_1(x)\ dx + \beta \int_a^b s_2(x)\ dx.$$

Proof. Recall that s_1 and s_2 simple means that they are the sum of indicator functions f_i and g_i:

$$s_1(x) = c_1 f_1(x) + \cdots + c_n f_n(x) = \sum_i c_i f_i(x)$$
$$s_2(x) = d_1 g_1(x) + \cdots + d_m g_m(x) = \sum_j d_j g_j(x)$$

where $f_i = \mathbb{1}(x \in [a_i, b_i])$ and $g_j = \mathbb{1}(x \in [\ell_j, k_j])$. Then $\alpha s_1(x) + \beta s_2(x)$ is also simple!

$$\alpha s_1(x) + \beta s_2(x) = \alpha c_1 f_1(x) + \cdots \alpha c_n f_n(x) + \beta d_1 g_1(x) + \cdots + \beta d_m g_m(x).$$

So by the definition of the integral of a simple function:

$$\begin{aligned}
\int_{x=a}^{b} \alpha s_1(x) + \beta s_2(x)\ dx &= \left[\sum_i \int_a^b \alpha c_i f_i(x)\ dx\right] + \left[\sum_j \int_a^b \beta d_j g_j(x)\ dx\right] \\
&= \left[\sum_i \alpha c_i (b_i - a_i)\right] + \left[\sum_j \beta d_j (k_j - \ell_j)\right] \\
&= \alpha \left[\sum_i c_i (b_i - a_i)\right] + \beta \left[\sum_j d_j (k_j - \ell_j)\right] \\
&= \alpha \left[\sum_i \int_a^b c_i f_i(x)\ dx\right] + \beta \left[\sum_j \int_a^b d_j g_j(x)\ dx\right] \\
&= \alpha \int_a^b s_1(x)\ dx + \beta \int_a^b s_2\ dx
\end{aligned}$$

□

- Suppose a simple function s lies below f
- Then want $\int_{x\in[a,b]} s(x)\ dx$ to be a lower bound for $\int_{x\in[a,b]} f(x)\ dx$. So

$$\int_{x\in[a,b]} s(x)\ dx \leq \int_{x\in[a,b]} f(x)\ dx.$$

- Similarly, if $r(x)$ is a simply function with $f(x) \leq r(x)$, then want

$$\int_{x\in[a,b]} f(x)\ dx \leq \int_{x\in[a,b]} r(x)\ dx.$$

Definition 17
Let $\mathcal{S}$ be the set of simple functions s such that $s < f$, and $\mathcal{R}$ be the set of simple functions r such that $f < r$. Then if there is a unique number I such that for all $s \in \mathcal{S}$ and $r \in \mathcal{R}$,

$$\int_{x\in[a,b]} s(x)\ dx \leq I \leq \int_{x\in[a,b]} r(x)\ dx,$$

then I is the **Riemann integral** of $f(x)$ over $[a,b]$, which is written $\int_a^b f(x)\ dx$.

Fact 13
$I = \int_{x\in[a,b]} f(x)\ dx$ exists iff there exists two sequences of simple functions $s_1, s_2, s_3, \ldots$ and $r_1, r_2, r_3, \ldots$ with $s_i \leq f \leq r_i$ for all i such that

$$\lim_{i\to\infty} \int_{[a,b]} s_i\ dx = \lim_{i\to\infty} \int_{[a,b]} r_i\ dx = I.$$

Theorem 1
If f is a continuous function over $[a,b]$, then the Riemann integral $\int_{x\in[a,b]} f(x)\ dx$ exists.

(Not going to prove this one: see Math Analysis I, Math 137.)

- Typically do not use this definition in calculating the integral, use it to prove facts about integrals.

Lemma 7
If $f(x)$ is integrable over $[a,b]$ and $c \in \mathbb{R}$, then $\int_{x\in[a,b]} c\cdot f(x)\ dx = c\int_{x\in[a,b]} f(x)\ dx$.

Proof. Let $I = \int_{x\in[a,b]} f(x)\ dx$. Let $\mathcal{S}$ be the set of simple functions s with $s \leq cf$, and $\mathcal{R}$ be the set of simple functions r with $cf \leq r$. Let $s \in \mathcal{S}$. Then s/c is a simple function with $s/c \leq f$. So $\int s/c \leq \int f = I$ Hence $\int s \leq cI$.

Similarly, for any $r \in \mathcal{R}$, $\int r \geq cI$.

So we have shown that cI is a number between the integral of any simple function bounding f below, and the integral of any simple function bounding f above. Is it unique?

Suppose J is such a bound. Then for all $s \in \mathcal{S}$, $\int s \leq J$, so $\int s/c \leq J/c$. Similarly for $r \in \mathcal{S}$, $\int r/c \geq J/c$. So $J/c = I$, which means $J = cI$. So yes, the integral of cf over $[a,b]$ is unique! □

This definition allows us to prove two more important rules about integrals:

Lemma 8
If f and g are integrable over $[a,b]$, then integration acts as a linear operator, so for constants c_1 and c_2,

$$\int_a^b c_1 f(x) + c_2 g(x)\ dx = c_1\int_a^b f(x)\ dx + c_2\int_a^b g(x)\ dx.$$

Lemma 9
If $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx.$$

Calculating integrals

- We don't use the definition to calculate integrals, so how are they calculated?
- One of the breakthrough realizations of humanity.
- Connects derivatives and integrals together.
- Fundamental theorem of Calculus: If $f = F'$ is continuous over $[a,b]$, then

$$\int_{x\in[a,b]} f(x)\ dx = F(b) - F(a).$$

9.1 A function with no Riemann integral

When does the Riemann integral exist?

- It always does when $f(x)$ is continuous over $[a, b]$
- It might not when $f(x)$ is not continuous
- Example over $[0, 1]$:
$$f(x) = \mathbb{1}(x \text{ has a finite decimal expansion}).$$
- So $f(0.324) = 1$, $f(1/3) = f(0.33333\ldots) = 0$.
- Note $0.69999\ldots = 0.7$, so $f(0.6999999\ldots) = f(0.7) = 1$.
- Now for any $a < b$, there exists c_1 and c_2 such that $a < c_1 < c_2 < b$, c_1 has a finite decimal expansion, and c_2 does not. Example
$$\begin{aligned} a &= 0.34523452 \\ b &= 0.34538249 \\ c_1 &= 0.34524 \\ c_2 &= 0.34524111\ldots \end{aligned}$$

10 Infinite Series

Question of the Day What is

$$1 + (1/3) + (1/3)^2 + (1/3)^3 + \cdots?$$

Today

- Infinite series
- Convergence and divergence

Recall

- A sequence is a function from $\{1, 2, \ldots, \}$ to $\mathbb{R}$.
- Example: $f(i) = 1/i^2$.
- Could also write as $1, 1/4, 1/9, 1/16, \ldots$
- Or $a_i = 1/i^2$
- Note: $\lim_{i\to\infty} a_i = 0$

Infinite series

- An infinite series (or just series for short) is the sum of the terms in a sequence
- Example:
  - $1 + 1/4 + 1/9 + \cdots$ is a series.
  - $1, 1/4, 1/9, 1/25, \ldots$ is a sequence.
- Qotd: $1 + (1/3) + (1/3)^2 + \cdots$ is a series.
- Now let's formalize this idea:

Definition 18
A **partial sum** of a sequence $\{a_i\}$ is $S_n = a_1 + a_2 + \cdots + a_n$, that is, the sum of the first n terms in the sequence.

Definition 19
A **series** (or infinite series) is the limit of the partial sums of a sequence as n goes to infinity. Write

$$\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i.$$

- It is not important what the number of first term is

$$\sum_{i=0}^{\infty} a_i \text{ or } \sum_{i=10}^{\infty} a_i$$

  are also considered series.
- Sometimes the limit of the partial sums is not a real number.

Definition 20
If $\sum_{i=1}^{\infty} a_i$ does not exist, then say the series **diverges**. If it does exist, say that the series **converges**.

Example

- $1+1+1+1+\cdots$ has partial sums $1, 2, 3, \ldots$ which diverges
- $1-1+1-1+1+\cdots$ has partial sums $1, 0, 1, 0, 1, 0, \ldots$ which diverges
- Note that divergences or convergence does not depend on where the series starts.

Lemma 10
Let $\{a_i\}_{i=0}^{\infty}$ be a sequence. Then either $\sum_{i=j}^{\infty} a_i$ converges for all $j \geq 0$ or $\sum_{i=j}^{\infty} a_i$ diverges for all $j \geq 0$.

- Later in the course we'll learn some tests to see whether a series converges or diverges. But for now we'll attack the problems directly.

10.1 Geometric series

- Qotd: What is $1 + (1/3) + (1/3)^2 + \cdots$?
- Let's looks at the partial sums for $a_i = (1/3)^i$. (Starts with $i = 0$)

n	S_n	approx S_n
0	1	1
1	$1 + 1/3 = 4/3$	1.3333333...
2	$4/3 + 1/9 = 13/9$	1.4444444...
3	$13/9 + 1/27 = 40/9$	1.481481...
$\vdots$	$\vdots$	
10	88573/59049	1.49999153...

$$1 = \frac{3}{2} - \frac{1}{2}, \quad \frac{4}{3} = \frac{3}{2} - \frac{1}{2} \cdot \frac{1}{3}, \quad \frac{13}{9} = \frac{3}{2} - \frac{1}{2} \cdot \frac{1}{9}, \quad \frac{88573}{59049} = \frac{3}{2} - \frac{1}{2} \cdot \frac{1}{3^{10}}$$

- Okay, so limit is probably 3/2, but how do you prove it?
- Need a way to calculuate the partial sums. A clever trick helps!

$$\begin{aligned} S_n &= 1 + (1/3) + (1/3)^2 + \cdots + (1/3)^n. \\ (1/3)S_n &= (1/3) + (1/3)^2 + \cdots + (1/3)^{n+1} \\ &= S_n - 1 + (1/3)^{n+1}. \end{aligned}$$

Now treat S_n as an unknown and solve this equation!

$$\begin{aligned} (1/3)S_n &= S_n - 1 + (1/3)^{n+1} \\ 1 - (1/3)^{n-1} &= S_n - (1/3)S_n = S_n(1 - 1/3) \\ S_n &= \frac{1 - (1/3)^{n+1}}{1 - 1/3}. \end{aligned}$$

- As $n \to \infty$, $(1/3)^{n+1} \to 0$, so $\lim_{n\to\infty} S_n = 1/(1-1/3) = 1/(2/3) = 3/2 = 1.500$.
- Used 1/3 here for illustration, but this trick works with any series of the form $1 + r + r^2 + r^3 + \cdots$ for $r \in \mathbb{R}$.
- I used r here for the variable because it is the *ratio* between successive terms in the sequence.
- These types of series come up a lot in geometry.

Definition 21
A series of the form $\sum_{i=0}^{\infty} r^i$ is called a **geometric series**.

Lemma 11
For $r \neq 0$,

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}.$$

Lemma 12
A geometric series converges if and only if $|r| < 1$. When $|r| < 1$,

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1 - r}.$$

(If and only if means that if $|r| \geq 1$, the series diverges.

Example

- Q: Which of the following series diverge?

$$\sum_{i=0}^{\infty} (1/2)^i, \quad \sum_{i=10}^{\infty} 3^i, \quad \sum_{i=1}^{\infty} (-1/5)^i.$$

- A: The second one diverges, the other two converge.
- What does $\sum_{i=1}^{\infty} (-1/5)^i = ?$
- A: We know $\sum_{i=0}^{\infty} (-1/5)^i = 1/(1 - (-1/5)) = 1/(6/5) = 5/6$.
  Since $\sum_{i=0}^{\infty} (-1/5)^i = 1 + \sum_{i=1}^{\infty} (-1/5)^i$, $\sum_{i=1}^{\infty} (-1/5)^i = 5/6 - 1 = -1/6 \approx -0.1666$.

10.2 The Harmonic Series

Definition 22
The series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}$$

is called the **harmonic series**

Lemma 13 (Harmonic series diverges)
The harmonic series sums to infinity.

Proof. Consider breaking the terms up into groups. The first group is the first term $1 \geq 1/2$. Then $1/2 + 1/3 \geq 1/4 + 1/4 = 1/2$ is the second group. The third group is $1/4 + 1/5 + 1/6 + 1/7 \geq 1/8 + 1/8 + 1/8 + 1/8 = 1/2$. So each group contains twice as many terms as the one before, which makes the sum of each group at least $1/2$. Therefore

$$1 + 1/2 + 1/3 + \cdots \geq 1/2 + 1/2 + 1/2 + 1/2 + \cdots = \infty.$$

$\square$

- The harmonic series is an important example
- Even through the terms of a series can be going to 0 in the limt, the sum of the terms can be going to infinity (diverging)

11 Derivatives: Inverses and Implicit

Question of the Day What is the derivative of $\arctan(x)$?

Today

- The inverse rule for derivatives.

11.1 Inverse functions

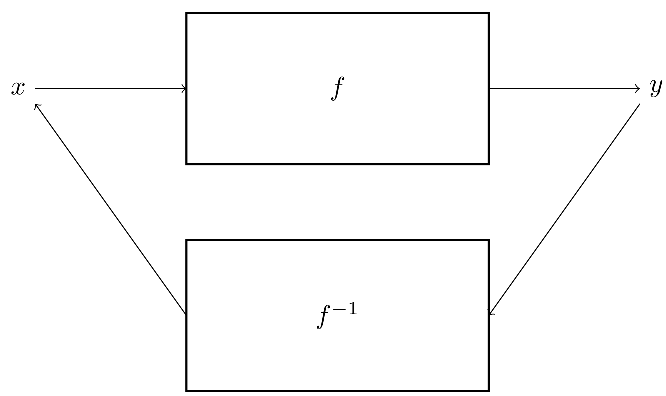
Definition 23
Let $f : A \to B$ be a function such that if $f(x) = f(y)$, then $x = y$. Such a function is called **one-to-one** (often written 1-1) or injective.

Definition 24
Let $f : A \to B$ be a function such that for all $y \in B$, there exists an $x \in A$ such that $f(x) = y$. (So $(\forall y \in B)(\exists x \in A)(f(x) = y)$.) Call such a function f **onto** or surjective.

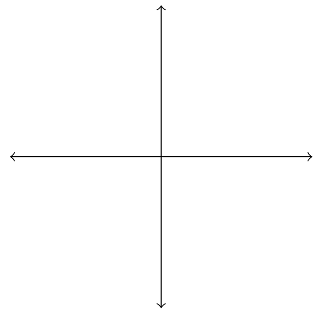
Definition 25
If $f : A \to B$ is both 1-1 and onto, then it has an **inverse function** $f^{-1} : B \to A$ such that for all $x \in A$, $f^{-1}(f(x)) = x$.

Lemma 14
If f^{-1} is the inverse of f, then f is the inverse of f^{-1}. So $[f^{-1}]^{-1} = f$.

- If $y = f(x)$, then f takes input x and outputs y. The inverse function f^{-1} takes input y and outputs x.



- Note: while $f^2(x)$ commonly means take f and raise to the nth power, f^{-1} is the exception, this does **not** mean raise f to the -1 power.
- Example: $\exp : \mathbb{R} \to (0, \infty)$ has inverse $\ln : (0, \infty) \to \mathbb{R}$. $\ln(\exp(x)) = x$
- To graph a function and its inverse, reflect the function over the line $y = x$:



Inverses and derivatives

- Remember the derivative of f is the ratio between the change in y and the change in x,
$$f'(x) = \frac{dy}{dx}.$$
- With inverses, just change x and y:
$$[f^{-1}(x)]' = \frac{dx}{dy}.$$
- One of those theorems that says treat dx and dy like variables instead of differentials: $dx/dy = 1/(dy/dx)$.

Lemma 15 (The Inverse Rule)
Suppose $f : A \to B$ is invertible and differentiable. Then
$$[f^{-1}]'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Easier to use in practice then statement might look like!

Example

- $y = x^2$ (function from $[0, \infty) \to [0, \infty)$)
- $x = \sqrt{y}$
$$\frac{dy}{dx} = 2x, \quad \frac{dx}{dy} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$$
- Matches power rule:
$$[x^{1/2}]' = (1/2)x^{(1/2)-1} = 1/(2x^{1/2} = 1/(2\sqrt{x})$$

Example

- Given that $[\ln(t)]' = 1/t$, what is $[e^x]'$?
- $$y = e^x, \quad x = \ln(y), \quad \frac{dx}{dy} = \frac{1}{y} = \frac{1}{e^x} \Rightarrow \frac{dy}{dx} = e^x.$$

Example

- What is $[\arctan(x)]'$?
- Let $t = \tan(\theta) = \sin(\theta)/\cos(\theta) = \sin(\theta)[\cos(\theta)]^{-1}$ Then

$$\begin{aligned}
\frac{dt}{d\theta} &= \sin(\theta)[[\cos(\theta)]^{-1}]' + [\sin(\theta)]'[\cos(\theta)]^{-1} && \text{Product Rule} \\
&= \sin(\theta)(-1)[\cos(\theta)]^{-2}[\cos(\theta)]' + [\sin(\theta)]'[\cos(\theta)]^{-1} && \text{Power and chain Rules} \\
&= -\frac{\sin(\theta)\cdot(-\sin(\theta))}{\cos^2(\theta)} + \frac{\cos(\theta)}{\cos(\theta)} && \text{Sin and cos derivatives} \\
&= 1 + \tan^2(\theta) = \sec^2(\theta) = \frac{1}{\cos^2(\theta)}. \\
\frac{d\theta}{ds} &= \frac{1}{1+\tan^2(\theta)} = \frac{1}{1+s^2}.
\end{aligned}$$

- So $[\arctan(x)]' = 1/(1+x^2)$.

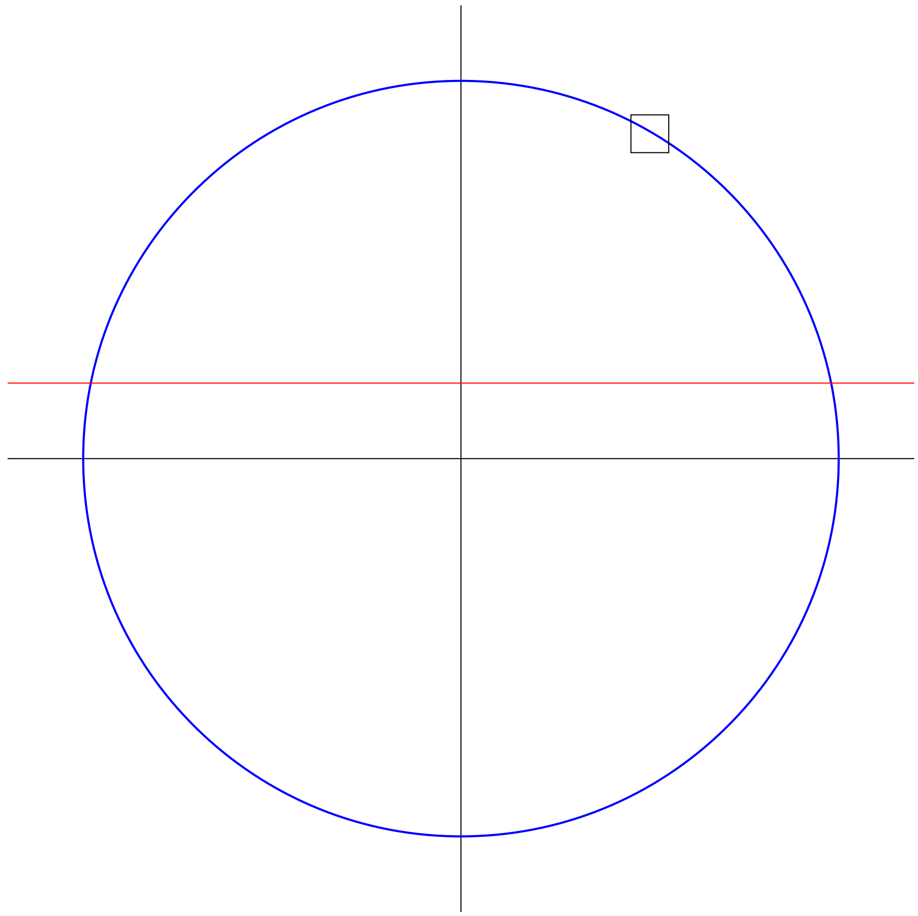
Lemma 16 (Arctan derivative)
The derivative of $\arctan(x)$ is $1/(1+x^2)$.

11.2 Implicit differentiation

- More general than inverses is idea of implicit differentiation
- Recall the equation describing points on a circle:

$$x^2 + y^2 = 1.$$

- This isn't a function exactly, since graph hits horizontal lines in two places



- But if we restrict ourselves to a small region around some points, it is invertible.
- What is dy/dx (or dx/dy) at $(x, y) = (1/2, \sqrt{3}/2)$.
- Start with equality of the circle
$$x^2 + y^2 = 1$$
- Differentiate term by term with respect to x
$$\begin{aligned} [x^2 + y^2]' &= [1]' \\ [x^2]' + [y^2]' &= 0 \\ 2x + 2y[y]' &= 0 \end{aligned}$$
  Now solve for y'!
$$y' = -\frac{2x}{2y} = -x/y$$
  and at $(1/2, \sqrt{3}/2)$, $y' = -(1/\sqrt{3}) \approx -0.5773502\ldots$.
- This technique of solving for y' is called *implicit differentiation*

Example

- What is y' in terms of x and y if $xy = 1$?
$$\begin{aligned} [xy]' &= [1]' \\ xy' + x'y &= 0 \\ xy' + y &= 0 \\ y' &= -y/x \end{aligned}$$

- Reciprocal of what we got with the circle!
- $xy = 1$ is a hyperbola and $x^2 + y^2 = 1$ is a circle.

12 Proving Derivative Rules

Question of the Day Give a formal proof of the chain rule

Today

- Formal proof of the chain rule

12.1 Proof of the chain rule

- If $y = f(x)$ and $z = g(y)$ are differentiable,

$$[g \circ f]' = [g' \circ f] \cdot f'$$
$$[g \circ f]'(x) = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = g'(f(x)) \cdot f'(x).$$

- Recall the intuition: if x changes then y changes which causes z to change. then

$$(dz/dx) = (dz/dy) \cdot (dy/dx) = g'(y)f'(x) = g'(f(x))f'(x).$$

- Now let's make this intuition precise!

Proof of Lemma 4. The derivative we want is

$$\lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h}.$$

Fix x, then $y = f(x)$. It will help to define new functions

$$w(k) = \frac{g(y+k) - g(y)}{k} - g'(y)$$
$$v(h) = \frac{f(x+h) - f(x)}{h} - f'(x)$$

By the defn of derivatives, $\lim_{k \to 0} w(k) = 0$ and $\lim_{h \to 0} v(h) = 0$.

Rearranging gives

$$g(y+k) = g(y) + [w(k) + g'(y)]k$$
$$f(x+h) = f(x) + [v(h) + f'(x)]h$$

Now apply g to both sides of the $f(x+h)$ equation to get

$$\begin{aligned} g(f(x+h)) &= g(f(x) + (v(h) + f'(x))h) \\ &= g(y + (v(h) + f'(x))h) \\ &= g(y+k) \end{aligned}$$

where we have set $k = (v(h) + f'(x))h$. Expanding $g(y+k)$ then gives:

$$\begin{aligned} g(f(x+h)) &= g(y) + [w(k) + g'(y)]k \\ &= g(f(x)) + [w(k) + g'(y)][v(h) + f'(x)]h. \end{aligned}$$

Bring the $g(f(x))$ over to the left hand side and divide by h to get:

$$\frac{g(f(x+h)) - g(f(x))}{h} = [w(k) + g'(y)][v(h) + f'(x)].$$

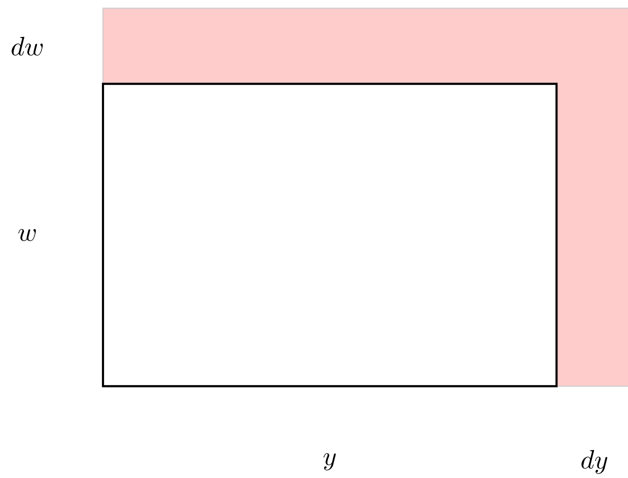
Note that $\lim_{h \to 0} k = 0$ since $v(h) \cdot h \to 0$ and $f'(x) \cdot h \to 0$. So as $h \to 0$, $w(k) \to 0$ and $v(h) \to 0$. Hence

$$\lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h} = [g'(y)][f'(x)] = g'(f(x))f'(x).$$

$\square$

12.2 Proof of the product rule

- The rule: $[fg]' = f'g + fg'$.
- The picture: $y = f(x)$, $w = g(x)$,



- Old area yw, new area $(y + dy)(w + dw)$
- So

$$\begin{aligned} d(yw) &= (y + dy)(w + dw) - yw \\ &= yw + w\ dy + y\ dw + (dy)(dw) - yw \\ &= w\ dy + y\ dw \end{aligned}$$

- Divide through by dx to get
$$\frac{d(yw)}{dx} = w\frac{dy}{dx} + y\frac{dw}{dx} = f'g + fg'.$$
- Let's turn this idea into a formal proof...

Proof of Lemma 2. Let's turn $d(yw) = w\ dy + y\ dw$ into a functional statement. Use $w = g(x + h)$, $dw = g(x + h) - g(x)$, $y = f(x)$, $dy = f(x + h) - f(x)$. Note that to turn this into a proof, we use $w = g(x + h)$ rather than $g(x)$ because as $h \to 0$, these will be the same.

$$\begin{aligned} &g(x + h)[f(x + h) - f(x)] + f(x)[g(x + h) - g(x)] \\ &\qquad = g(x + h)f(x + h) - g(x + h)f(x) + f(x)g(x + h) - g(x)f(x) \\ &\qquad = g(x + h)f(x + h) - g(x)f(x). \end{aligned}$$

Now divide by h and take the limit as $h \to 0$. The left hand side becomes:

$$\begin{aligned} &\left[\lim_{h\to 0} g(x+h)\right]\left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] + f(x)\left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right] \\ &\qquad = g(x)f'(x) + f(x)g'(x). \end{aligned}$$

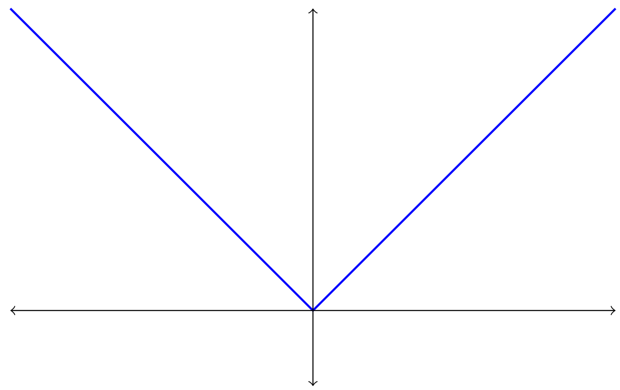
The right hand side becomes:

$$\lim_{h\to\infty} \frac{g(x+h)f(x+h) - g(x)f(x)}{h} = [f(x)g(x)]'$$

thus proving the product rule. $\square$

12.3 Differentiable functions are continuous

- A function with a finite derivative is continuous
- Intuition: when dx small $dy = f'(x)dx$ so also small
- There are functions that are continuous where the derivative does not exist
- Canonical example: $f(x) = \text{abs}(x) = |x|$ at $x = 0$.



- Is slope of tangent line $+1$ or -1? Could be either!

Lemma 17
A function with finite derivative $f'(a)$ is continuous at a.

Proof. Let $w(h) = [f(a+h) - f(a)]/h - f'(a)$. Then using the composition rule of limits with $h = x - a$ gives

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{h\to 0} f(a+h)\\ &= \lim_{h\to 0} w(h) + f'(a)h + f(a) = f(a).\end{aligned}$$

$\square$

13 Sets operations, closed and open, supremum and infimum

Question of the Day What makes $A \subset \mathbb{R}$ closed?

Today

- What makes a closed set closed
- More set notation
- Open set characterization of continuity
- Suprema and infima

13.1 Closed and open sets

- $[a, b] = \{x : a \leq x \leq b\}$ is a closed interval
- $(a, b) =]a, b[= \{x : a < x < b\}$ is an open interval
- $(a, b] = \{x : a < x \leq b\}$ is neither open nor closed
- So what do these terms mean?
- Euclidean distance: means distance between two points on a line

$$\text{dist}_{\text{Euclidean}}(x, y) = |x - y|.$$

Definition 26
A set $A \subset \mathbb{R}$ is **open** (under Euclidean distance) if

$$(\forall x \in A)(\exists \epsilon > 0)((x - \epsilon, x + \epsilon) \subseteq A).$$

Example

- The interval $(0, 1)$ is open.
- If $x = 0.1$, set $\epsilon = 0.05$, $(0.05, 0.15) \subseteq A$.
- Proof: Let $x \in (0, 1)$. Set $\epsilon = \min\{x/2, (1 - x)/2\}$, then $(x - \epsilon, x + \epsilon) \subseteq (\epsilon, 1 - \epsilon) \subseteq (0, 1)$ $\square$

Definition 27
The **complement** of a set is all elements that are not in the set. Write $A^C = \{x : x \notin A\}$.

Example

- $A = [0, 1]$, $A^C = \{x : x < 0 \text{ or } x > 1\}$.

Definition 28
A set is **closed** if its complement is open.

13.2 Union and Intersection

- Notation for "or" is useful
- Mathematical or means one or other or both are true.
- $(5 > 3)$ or $(3 > 1)$ is true, $(5 > 3)$ or $(1 > 3)$ is true, $(3 > 5)$ or $(1 > 3)$ is false.

Definition 29
The **union** of two sets is all elements that are in one set or the other or both. Write $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Example: $[a, b] = (-\infty, a) \cup (b, \infty)$

- In mathematics, "and" means both things are true
- $(5 > 3)$ and $(3 > 1)$ is true, $(5 > 3)$ and $(1 > 3)$ is false.

Definition 30
The **intersection** of two sets is all elements that are in both sets. Write $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Example: $[3, 5] \cap [1, 4] = [3, 4]$

13.3 Closed under limits

- Mathematicians used "closed" in general to mean that set is closed under an operation
- For instance, the positive integers $\{1, 2, 3, \ldots\}$ are closed under addition. Take any two positive integers and add them, and you get a positive integer.
- The positive integers are *not* closed under subtraction.
- All integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are closed under subtraction.
- All integers are *not* closed under division.
- In this way of looking at things, saying an interval (or set) is closed is shorthand for saying that it is closed under limits.

Lemma 18
Suppose for all $i \in \{1, 2, \ldots\}$, $a_i \in A$, and $\lim_{i \to \infty} a_i = L$ exists. If A is closed, then $L \in A$.

Examples

- $0.9, 0.99, 0.999, \ldots$ converges to 1
- $[0, 1]$ is a closed set, so $1 \in [0, 1]$
- $(0, 1)$ is not closed, because $1 \notin [0, 1)$
- The set $(0, 1]$ is not closed, but this sequence does not prove it!
- The sequence $0.1, 0.01, 0.001, \ldots \to 0$ does prove that $(0, 1]$ is not closed.

13.4 Supremum and Infimimum

- Two more concepts related to limits
- The supremum is the least upper bound on a set of numbers.
- For instance, the set $\{1, 2, 3\}$ has an upper bound of 14, 11, 4, and 3.
- Can't make the upper bound any smaller than 3. So 3 is the least upper bound, write $\sup(\{1, 2, 3\}) = 3$.
- With finite sets, supremum=maximum.
- Advantage of supremum is that *all* subsets of real numbers have a supremum.
- Example: $\sup(\{0.9, 0.99, 0.999, \ldots\}) = 1$
- Example: $\sup(\{1, 2, 3, \ldots\}) = \infty$
- Example: $\sup(\{\}) = \sup(\emptyset) = -\infty$. Why: since any number is an upper bound on nothing, all real numbers are upper bounds. The smallest number of all "real" numbers is $-\infty$.
- Why we learn about supremum: need it to prove the Intermediate Value Theorem

Definition 31
The **extended real numbers** are $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

- Informally, supremum=least upper bound
- Formally, the supremum is defined as follows.

Definition 32
The **supremum** of $S \subseteq \mathbb{R}$ is defined as follows.

1: $\sup(S) = -\infty$ if S is empty

2: If there exists a b such that $S \subseteq (-\infty, b]$, then $\sup(S) = \min\{b : S \subseteq (-\infty, b]\}$.

3: $\sup(S) = \infty$ if $(\forall n)(\exists s \in S)(s > n)$

Mnemonic for supremum

- Superman flies above the buildings, but as low as possible so he can respond to crime quickly

Infimum

- The opposite of supremum, infimum is the greatest lower bound.
- Same root as inferior, infernal, lies below the set S

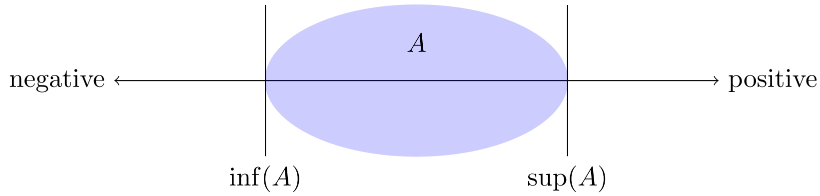
Definition 33
The **infimum** of $S \subseteq \mathbb{R}$ is

1: ∞ if S is empty

2: $\max\{a : S \subseteq [a, \infty)\}$ if such an $a \in \mathbb{R}$ exists

3: $-\infty$ if $(\forall x)(\exists s \in S)(s < x)$

Lemma 19
Let $-S = \{x : -x \in S\}$. Then $\sup(S) = -\inf(-S)$.



14 The Intermediate Value Theorem

Question of the Day Is there a root of $x^3 - 2x + 3$ between -2 and -1.5?

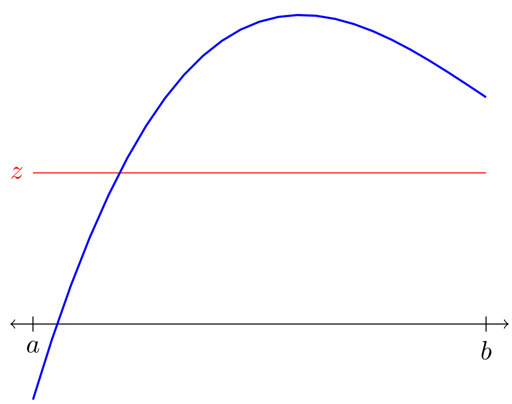
Today

- The Intermediate Value Theorem
- Supremum and infimum
- The Bisection method

Qotd

- A *root* is a place where a function equals 0
- Here $f(x) = x^3 - 2x + 3$
- $f(-2) = -8 + 4 + 3 = -1$
- $f(-3/2) = -27/8 + 3 + 3 = 21/8$
- So at -2 function is below 0, at $-3/2$ it is above zero
- Since f is continuous, intuition is that at some point, it must have crossed 0
- Intuition true! Result called Intermediate Value Theorem

Theorem 2 (Intermediate Value Theorem)
Let f be a continuous function over $[a, b]$. For all z between $f(a)$ and $f(b)$, there exists a c such that $f(c) = z$



History

- Definitions in mathematics are created by mathematicians, not discovered written in stone
- Some thought that this should be the defining notion of continuity
- There are functions that satisfy Intermediate Value Theorem that do not satisfy limit notion of continuity
- Example: $f(x) = \sin(1/x)$ when $x \neq 0$, $f(0) = 0$
- $\lim_{x \to 0} f(x)$ does not exist, so not continuous at 0.

14.1 Bisection Algorithm

- Goal: solve $f(x) = 0$
- Suppose $f(a)f(b) < 0$ (so $\min\{f(a), f(b)\} < 0$, $\max\{f(a), f(b)\} > 0$)
- Then let $c = (a+b)/2$
- If $f(a)f(c) > 0$, then let $a \leftarrow c$
- Else $b \leftarrow c$
- Repeat until a is close to c

Example: $f(x) = x^3 - 2x + 3$

a	b	c	$f(a)$	$f(b)$	$f(c)$
-2	-1.5	-1.75	-1	2.625	1.140625
-2	-1.75	-1.875	-1	1.140625	0.158203125
-2	-1.875	-1.9375	-1	0.158203125	-0.398193359

So the root is in $[-1.9375, 1.875]$

14.2 Proof of IVT

Idea of the proof

- Let z be any value between $f(a)$ and $f(b)$.
- There is at least one value x such that $f(x) \geq z$
- Let $c = \sup\{x : f(x) \geq z\}$
- If f is continuous, then $f(c) = z$
- A logic fact will come in handy here...

  Lemma 20 (Contrapositive rule)
  p implies q is true if and only if $\neg q$ implies $\neg p$ is true.

  Example: If an animal is a bear, then it is a mammal. If an animal is not a mammal, then it is not a bear. This is called the contrapositive.

- Before proving the theorem, helpful to know a little more about suprema

Lemma 21
Suppose $s = \sup(S)$ and $s \in (-\infty, \infty)$. Then

$$(\forall \epsilon > 0)(\exists s' \in S)(s - s' \in [0, \epsilon))$$

Proof. Let $\epsilon > 0$. Let b be a real number such that $S \subseteq (-\infty, b]$. Then $s \leq b$. If $S \subseteq (-\infty, b - \epsilon]$, then $s \leq b - \epsilon$. So if $s > b - \epsilon$, then $S \cap (b - \epsilon, b]$ is nonempty. Let s' be any element of S in $(b - \epsilon, b]$. Then $s' \leq s$ since s is a supremum so $s' \in (b - \epsilon, s] \subseteq (s - \epsilon, s]$. □

Proof of Theorem 2. Without loss of generality, assume $f(a) \leq f(b)$. Otherwise consider $g(x) = f(a+(b-x))$.

Let $z \in [f(a), f(b)]$. Then if $z = f(a)$, or $z = f(b)$, then we are done!

Otherwise, let $c = \sup\{x \in [a, b] : f(x) \geq z\}$. Then since $f(b)$ is at least z, $c \leq b$. Since $c \in [a, b]$, $c \geq a$.

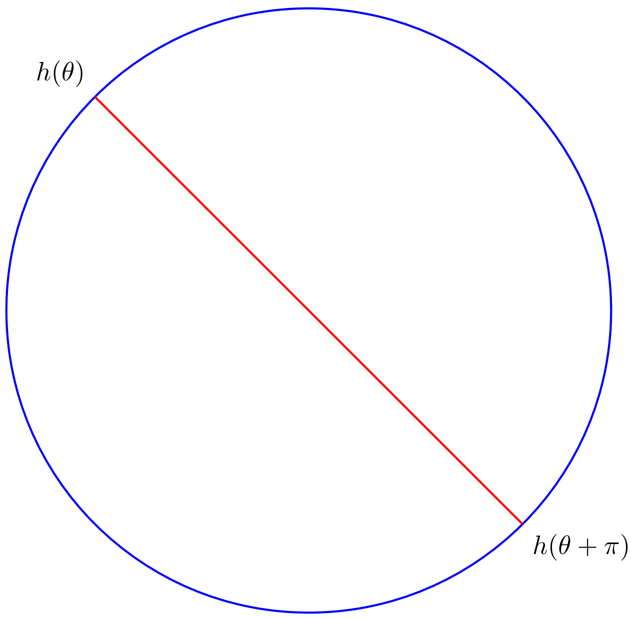
Let c' be any point in (a, c') with $f(c') - z \neq 0$. Let $\epsilon = |f(c') - z|/2$. Then since f is continuous, there exists δ such that for all $x \in [a, b] \cap [c' - \delta, c' + \delta]$, $|f(x) - z|$ □

14.3 Weird applications of the Intermediate Value Theorem

Traveling around a circle

- Suppose you travel in a circle
- Each point on the circle has its own height (at least 0)

> **Fact 14**
> If the height of the circle as a function of angle is continuous, then there must be two points on the circle somewhere that are opposite one another with the same height.



Proof. If θ is an angle, $\theta + \pi$ is the angle on the opposite side of the circle.

Let $f(\theta) = h(\theta) - h(\theta + \pi)$ be the difference between the two points on the circle. Then if $f(0) = 0$ we are done. Suppose $f(0) \neq 0$. Then $f(\pi) = h(\pi) - h(2\pi) = -[h(0) - h(\pi)] = -f(0)$. In other words, $f(\pi)$ and $f(0)$ have opposite sign. So there must be an angle θ in $(0, \pi)$ with $f(\pi) = 0$. That means $h(\theta) = h(\theta + \pi)$. □

Round trips

A similar result is the following:

> **Fact 15**
> Suppose I take a round trip on continuous terrain. Then if I do not start at the highest or lowest point on the path, at some point in my trip I will be exactly as high as when I started.

Proof. Let $f(t)$ be the height I start at time 0 minus the height I am at time t. If I take T time to complete the loop, $f(T) = f(0) = 0$. Some point t_1 has $f(t_1) > 0$ since I didn't start at the maximum, and some point t_2 has $f(t_2) < 0$ since I didn't start at the minimum. Between t_1 and t_2 must be another point t_3 with $f(t_3) = 0$! □

15 First and Second derivatives

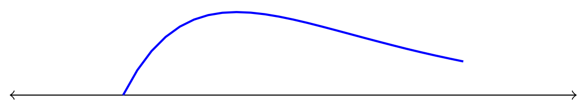
Question of the Day For $x \geq 0$, where is $x\exp(-x)$ increasing?

Today

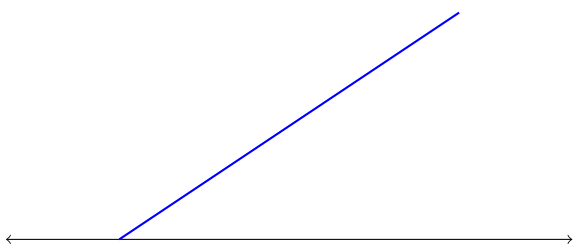
- First derivatives and increasing/decreasing
- Second derivatives and convex up and convex down

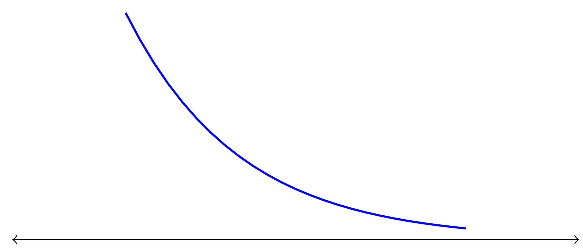
QotD

- The graph of the curve looks like



- Look at the factors x and $\exp(-x)$





- x wants to make it increase, $\exp(-x)$ wants to make it decrease
- Looking at graph, growth of x dominates at beginning, later on $\exp(-x)$ factor dominates.
- Which effect is smaller when?

Using derivatives

- Idea: Function is increasing when tangent line has positive slope
- Otherwise function is decreasing

Lemma 22
If f has a nonnegative derivative over (a, b), then f is increasing on $[a, b]$. If f has a nonpositive derivative over (a, b), then f is decreasing on $[a, b]$.

- Proof idea
- Recall that $f(x + h) = f(x) + f'(x)h + o(h)$.
- So for h smaller than H, $|o(h)| \leq (1/2)f'(x)h$
- So for all $h \in [0, H]$,

$$f(x + h) > f(x) + f'(x)h - (1/2)f'(x)h = f(x) + (1/2)f'(x)h.$$

- So if $f'(x) > 0$, all points c in $(x, x + H]$ have $f(c) > f(x)$
- If $f'(x) < 0$, all points c in $(x, x + H]$ have $f(c) < f(x)$

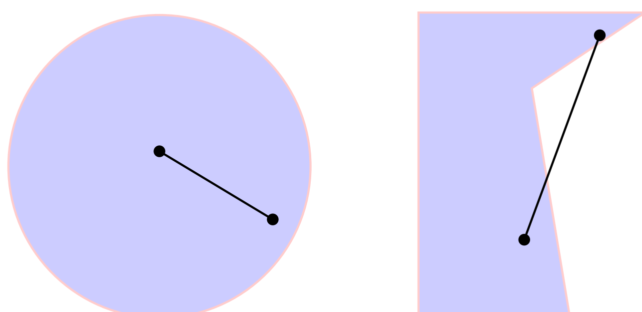
- Qotd: Where is $f(x) = x\exp(-x)$ increasing?
- Product rule:

$$\begin{aligned}[x\exp(-x)]' &= x[\exp(-x)]' + [x]'\exp(-x) \\ &= x(-\exp(-x)) + \exp(-x) \\ &= \exp(-x)(1-x)\end{aligned}$$

- Product positive when $x < 1$, negative when $x > 1$.
- When derivative is strictly positive, function is strictly increasing.

Lemma 23
If f has a positive derivative over (a, b), then f is strictly increasing on $[a, b]$. If f has a negative derivative over (a, b), then f is decreasing on $[a, b]$.

15.1 Convexity



convex   not convex

Definition 34
A set S is **convex** if for any two points in the set, the line segment that connects the two points is also in the set.

$$(\forall a, b \in S)(\forall \lambda \in [0, 1])(a\lambda + (1-\lambda)b \in S)$$

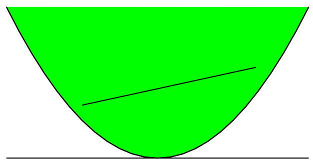
- When $\lambda = 0$, $a\lambda + (1-\lambda)b = b$
- When $\lambda = 1$, $a\lambda + (1-\lambda)b = a$
- When $\lambda = 1/5$, $a\lambda + (1-\lambda)b = (a+b)/2$

Definition 35
The **epigraph** of a function is those points that lie on or above the function of the graph.

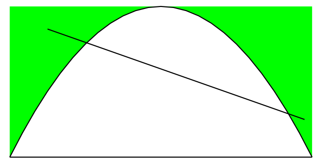
$$\text{epi}(f) = \{(x, y) : y \geq f(x)\}$$

- epi means “upon” “on” or “over” in Greek
- The epidermis are cells on top of the skin
- The epigraph lies on top of the graph

Definition 36
A function f is **convex** if its epigraph is convex.

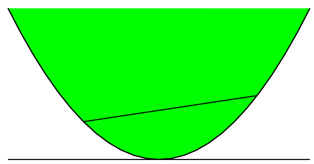


convex



not convex

- Need A to be convex so that $\lambda a + (1-\lambda)b \in A$ as well.
- Only need to check line segments that touch the graph



convex

Fact 16
The function x^2 is convex.

Proof. Let $a < b$ be real numbers. Let $\lambda \in [0, 1]$
Start with $(a-b)^2 > 0$. That gives $a^2 - 2ab + b^2 > 0$ so $a^2 + b^2 > 2ab$.
So

$$\begin{aligned} \lambda(1-\lambda)a^2 + \lambda(1-\lambda)b^2 &> 2\lambda(1-\lambda)ab \\ \lambda a^2 + (1-\lambda)b^2 &> \lambda^2 a^2 + (1-\lambda)^2 b^2 + 2\lambda(1-\lambda)ab \\ &= (\lambda a + (1-\lambda)b)^2. \end{aligned}$$

$\square$

- Complicated!
- A simpler way...
- Convex functions have increasing derivative
- $f'(x)$ is increasing if $f''(x)$ is nonnegative

Lemma 24 (Nonnegative second derivative implies concavity.)
Suppose that f has a continuous second derivative. Then f is convex over any interval where $f''(x) \geq 0$.

- $f(x) = x^2$ is convex
- $f'(x) = 2x$ is convex
- $f''(x) = 2 > 0$
- Example: Show that e^{-x} is convex
- $[e^{-x}]' = e^{-x}[-x]' = -e^{-x}$
- $[e^{-x}]'' = [-e^{-x}]' = -[e^{-x}]' = -(-e^{-x}) = e^{-x} > 0$
- So the original function is convex!

15.2 Concavity

Definition 37
A function f is **concave** if $-f$ is convex.

- Example: Show $-x^2$ is concave
- $-(-x^2) = x^2$ which is convex, so $-x^2$ is concave

15.3 C^n notation

- Convexity proof requires that f have continuous second derivative.
- Say that $f \in C^2$, or f is in C^2.

Definition 38
Say that $f \in C^n$ if the nth derivative of f is continuous. (C^0 is the class of continuous functions.) Say that $f \in C^n[a, b]$ if the nth derivative of f is continuous over the interval $[a, b]$. C^∞ is the class of functions for which all derivatives are continuous.

Definition 39
Say that $f \in C^n[a, b]$ if $f : [a, b] \to B$ and $f \in C^n$.

- Example: $|x| \in C^0$, but $|x| \notin C^1$
- Example: $e^x \in C^\infty$.

16 The Extreme Value Theorem

Question of the Day What is the maximum value of $f(x) = x\exp(-x)$ for $x \in [0, 3]$?

Today

- Boundedness theorem
- Extreme value theorem

Closed bounded intervals

- Closed bounded intervals have special properties

Definition 40
A set of real numbers A is **bounded** if there exists finite M such that $|a| \leq M$ for all $a \in A$.

- Example: $[-3, 4]$ is bounded, $(-\infty, 2)$ or $[0, \infty)$ are not.

Definition 41
An interval that is both closed and bounded is called **compact**.

- When an interval is not compact, continuous functions can fly off to infinity (or negative infinity)
- Example: $f_1(x) = 1/x$ for $x \in (0, \infty)$ can be arbitrarily large
- Example: $f_2(x) = x$ for $x \in [1, \infty)$ can be arbitrarily large
- Both these examples have bounded range over compact intervals though.

Theorem 3 (Boundedness Theorem)
A continuous function over a compact set has a range that is bounded.

- Continuous: draw without lifting pencil from the paper
- Closed: no asymptotic limits can happen
- Bounded: can't keep drawing forever.
- Since it is bounded above, the supremum of the range exists and is finite
- Since it is bounded below, the infimum of the range exists and is finite

Theorem 4 (Extreme Value Theorem)
Let f be continuous over $[a, b]$. Then

$$(\exists c, d \in [a, b])(\forall x \in [a, b])(f(c) \leq f(x) \leq f(d)).$$

That is, there exists c and d in $[a, b]$ such that $\max_{x \in [a,b]} f(x) = f(c)$ and $\min_{x \in [a,b]} f(x) = f(d)$.

- In other words, there exists a point where the function f is maximized and where it is minimized.

16.1 Optimization with the extreme value theorem

- Where can the maximum and minimum be?
- If f has a continuous derivative, then anywhere $f'(x) > 0$, f is strictly increasing, can't be a max or min...
- ...anywhere $f'(x) < 0$, f is strictly decreasing, can't be a max or min.
- Max or min can only be at the endpoints of the interval or places where $f'(x) = 0$.

Definition 42
A **critical point** is any place where $f'(x) = 0$.

Optimizing continuous f over $[a, b]$

1: Find the critical points

2: Evaluate f at the critical points, and at a and at b

3: The smallest function value must be the maximum of the function

4: The largest function value must be the minimum.

Qotd

- Find the critical points:

$$\begin{aligned}[x\exp(-x)]' &= [x]'\exp(-x) + x[\exp(-x)]' = \exp(-x) - x\exp(-x) \\ &= (1-x)\exp(-x).\end{aligned}$$

- Recall, if $rs = 0$, either $r = 0$ or $s = 0$. So if $(1-x)\exp(-x) = 0$, either $1 - x = 0$ or $\exp(-x) = 0$. Can't have $\exp(-x) = 0$, so only critical point is at $x = 1$.
- Make a table:

x	$f(x)$
0	0
1	$\exp(-1) \approx 0.3678\ldots$
3	$3\exp(-3) \approx 0.1493\ldots$

- Hence

$$\max_{x\in[0,3]} f(x) = 0.3678\ldots$$

- Not part of question, but also get:

$$\arg\max_{x\in[0,3]} f(x) = 1$$

and

$$\min_{x\in[0,3]} f(x) = 0, \quad \arg\min_{x\in[0,3]} f(x) = 0$$

Symbolic problems

- Nowadays software (WA) can do max and min numerically
- Interesting problems symbolic
- Example: Find the maximum value of $f(x) = \lambda^2 x \exp(-\lambda x)$ for $x \in [0, 2/\lambda]$, where $\lambda > 0$ is a constant.
- Find the critical points.

$$\begin{aligned}
[\lambda^2 x \exp(-\lambda x)]' &= \lambda^2 [x \exp(-\lambda x)]' \\
&= \lambda^2 [[x]' \exp(-\lambda x) + x[\exp(-\lambda x)]'] \\
&= \lambda^2 (\exp(-\lambda x) + x \exp(-\lambda x)[-\lambda x]') \\
&= \lambda^2 (\exp(-\lambda x) - \lambda x \exp(-\lambda x)) \\
&= \lambda^2 \exp(-\lambda x)(1 - \lambda x)
\end{aligned}$$

Setting equal to 0 gives $1 - \lambda x = 0$, $1 = \lambda x$, $x = 1/\lambda$ is the only critical point.

- Plug in values

x	$f(x)$
0	0
$1/\lambda$	$\lambda \exp(-1)$
$2/\lambda$	$2\lambda \exp(-2)$

- Which is bigger, $\lambda \exp(-1)$ or $2\lambda \exp(-2)$?
- Since $\lambda > 0$ and $\exp(-1) > 2 \exp(-2)$, $\lambda \exp(-1)$.
- $\boxed{\max_{x \in [0, 2/\lambda]} \lambda^2 x \exp(-\lambda x) = \lambda / e.}$
- Can check answer using WA.

17 Optima on unbounded regions

Question of the Day What is the maximum of $f(x) = x\exp(-x)$ for $x \in [0, \infty)$?

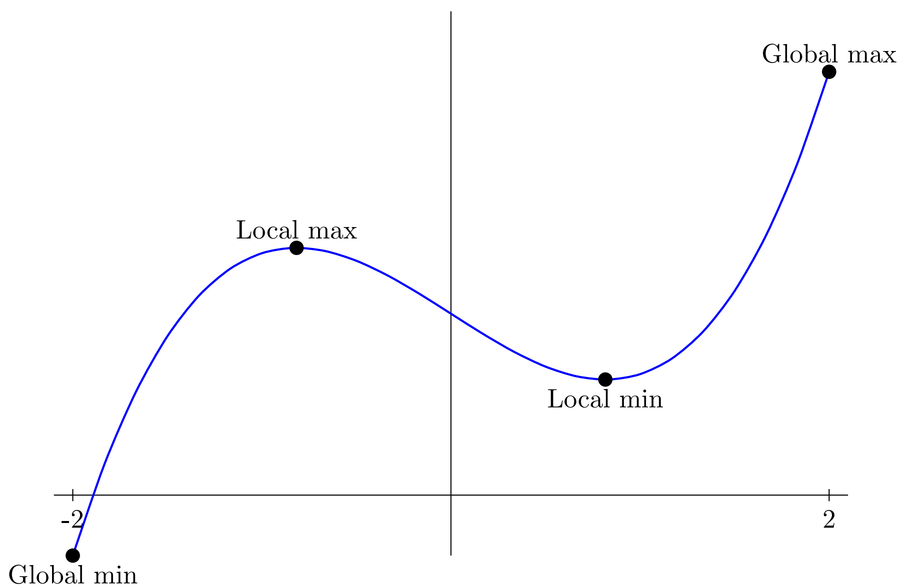
Today

- Optima (maxima and minima) over unbounded closed sets
- Global optima

Definition 43
Say that M is a **global maximum** for $f : A \to \mathbb{R}$ (write $M = \max_{x\in A} f(x)$) if there exists c such that $f(c) = M$, and for all $a \in A$, $f(a) \leq M$. Say that m is **global minimum** for $f : A \to \mathbb{R}$ (write $m = \min_{x\in A} f(x)$) if

$$(\exists d)(\forall a \in A)(f(a) \geq f(d) = m)$$

- Global maxima are maxima where the function value is as large as any output.
- Local maxima are maxima where the function value is largest in a neighborhood of the input value.



17.1 Local optima

Definition 44
Say (x, y) is a **local maximum** if $f(x) = y$, and there exists $[a, b]$ such that $x \in [a, b]$ and $f(x') \leq y$ for all $x' \in [a, b]$.

Definition 45
Say (x, y) is a **local minimum** if $f(x) = y$, and there exists $[a, b]$ such that $x \in [a, b]$ and $f(x') \geq y$ for all $x' \in [a, b]$.

- Global optima are also local optima
- To find local optima, just find points with $f'(x) = 0$, and try to find a narrow enough $[a, b]$ so that $f'(x)$ is the global optima for that region.

Example

- Show that $(1, -3)$ is a local minimum of $x^3 - 3x - 1$.
- For $f(x) = x^3 - 3x - 1$, $f'(x) = 3x^2 - 3$, so the critical points are at $3x^2 - 3 = 0$, so x is either -1 or 1.
- So $[0, 2]$ only contains the critical point at $x = 1$.
- $f(0) = -1$, $f(1) = -3$, $f(2) = 1$, so $(1, -3)$ is a local minima.

Using second derivatives

- Note that at a local minima, $f'(x) = 0$, for $a < x$ $f'(x) < 0$, and for $a > x$, $f'(x) > 0$. So $f'(x)$ is increasing, which means $f''(x) \geq 0$.

Lemma 25
Suppose $f \in C^2[a, b]$ and $x \in [a, b]$. If $f'(x) = 0$ and $f''(x) > 0$, then $(x, f(x))$ is a local minimum of the function f. If $f'(x) = 0$ and $f''(x) < 0$, then $(x, f(x))$ is a local maximum of the function f.

Example

- Should that $(1, -3)$ is a local minimum of $f(x) = x^3 - 3x - 1$
- $f'(x) = 3x^2 - 3$, $f''(x) = 6x$.
- $f(1) = 1 - 3 - 1 = -3$, $f'(1) = 3(-1)^2 - 3 = 0$, $f''(1) = 6(1) = 6 > 0$.
- So $(1, -3)$ is a local minimum

17.2 Optima over unbounded intervals

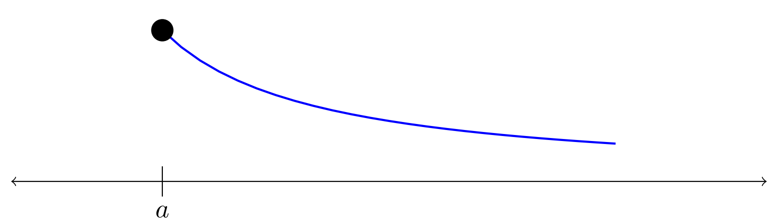
Now consider the problem of finding $\max_{x \in [a, \infty)} f(x)$.

- The following fact will be helpful.

Lemma 26
Suppose $f'(x) \leq 0$ for all $x \geq a$. Then
$$\max_{x \in [a, \infty)} f(x) = f(a).$$

Idea

- If $f'(x) \leq 0$, $f(x)$ is decreasing over $[a, \infty)$.
- Hence largest point is leftmost point



Proof. Note
$$\max_{x \in [a, \infty)} f(x) = f(a) \Leftrightarrow (\forall b > a)(f(b) \leq f(a)).$$
Let $b > a$. Since $f'(x) \leq 0$ for all $x \in [a, 2b]$, f is decreasing on $[a, 2b]$. So $f(b) \leq f(a)$. This shows that $(\forall b > a)(f(b) \leq f(a))$, which means $f(a) = \max_{x \in [a, \infty)} f(x)$. $\square$

Example

- Find $\max_{x\in[3,\infty)} x\exp(-x)$.
- As earlier, $[x\exp(-x)]' = \exp(-x)(1-x)$.
- For $x \geq 3$, $1-x<0$, $\exp(-x)>0$, so $f'(x) \leq 0$.
- Hence $\max_{x\in[3,\infty)} f(x) = f(3) = 3/e^3 = 0.1493\ldots$

Similar results hold for left intervals, max and min

Lemma 27
Suppose $f'(x) \geq 0$ over a closed set A. Then

1: $\max_{x\in A} = f(\max(A))$

2: $\min_{x\in A} = f(\min(A))$

Suppose $f'(x) \leq 0$ over a closed set A. Then

1: $\max_{x\in A} = f(\min(A))$

2: $\min_{x\in A} = f(\max(A))$

Don't memorize this lemma, draw the picture (as above)!

What if not increasing or decreasing over the region?

- Break the region into manageable pieces
- Do the optimization on each piece

Lemma 28
Suppose $\max_{x\in A} f(x)$ and $\max_{x\in B} f(x)$ exist. Then

$$\max_{x\in A\cup B} f(x) = \max\left\{\max_{x\in A} f(x), \max_{x\in B} f(x)\right\}.$$

Example

- What is $\max_{x\geq 0} x\exp(-x)$?
- Earlier found $\max_{x\in[0,3]} x\exp(-x) = 1/e$
- Also found $\max_{x\in[3,\infty)} x\exp(-x) = 3/e^3$. So

$$\begin{aligned}\max_{x\in[0,\infty)} x\exp(-x) &= \max\{1/e, 3/e^3\}\\ &= \max\{0.36\ldots, 0.14\ldots\} = 1/e = 0.3678\ldots.\end{aligned}$$

Example

- Find $\min_{x\leq 0} \frac{x}{x^2+2}$.
- To know where to break up $(-\infty, 0]$, helps to know critical points

- $f(x) = x/(x^2+2) = x(x^2+2)^{-1}$

$$\begin{aligned} f'(x) &= [x]'(x^2+2)^{-1} + x[(x^2+1)^{-1}]' \\ &= (x^2+2)^{-1} + x[(-1)(x^2+1)^{-2}[x^2+2]' \\ &= (x^2+2)^{-1} - 2x^2(x^2+1)^{-2} \end{aligned}$$

To solve $f'(x) = 0$, note that x^2+1 is always at least 2, so not equal to 0, so

$$\begin{aligned} 0 &= (x^2+2)^{-1} - 2x^2(x^2+1)^{-2} \\ 0 &= x^2+2-2x^2 \\ 0 &= 2-x^2 \\ x^2 &= 2 \\ x &\in \{-\sqrt{2}, \sqrt{2}\} \end{aligned}$$

Only worried about $x \leq 0$, so $-\sqrt{2}$.

- Recall $-\sqrt{2} \approx -1.41$. So break up into $[-2, 0]$ and $(-\infty, 2]$.

x	$f(x)$
-2	$-2/6 = -0.3333\ldots$
$-\sqrt{2}$	$-\sqrt{2}/4 \approx -0.3535\ldots$
0	0

Hence $\min_{x\in[-2,0]} x/(x^2+2) = -\sqrt{2}/4$

- Note $f'(x)$ is continuous, $f'(x) \neq 0$ in $(-\infty, 2]$ and $f'(-2) = -1/4$. So by the Intermediate Value Theorem, $f'(x) < 0$ for all $x \in (-\infty, 2]$. Hence $\min_{x\in(-\infty,-2]} x/(x^2+2) = f(-2) = -1/3$.

- Therefore $\min_{x\in(-\infty,-2]} f(x) = \min\{-\sqrt{2}/4, -1/3\} \approx \boxed{\text{-0.3535}}$

18 The Fundamental Theorem of Calculus

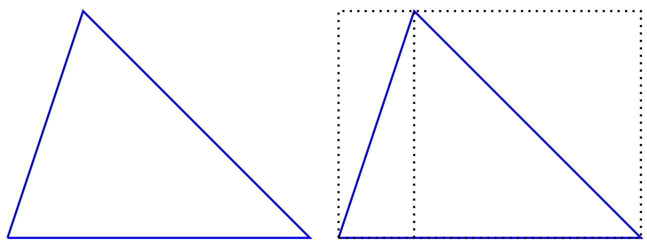
Question of the Day What is $\int_{x=0}^{2} x^3\ dx$?

Today

- The Fundamental Theorem of Calculus
- Intuition of FTC
- Second form of FTC

Area of a triangle by rectangle

- There is a trick in finding the area of a triangle
- Turn it into an area of a rectangle problem



- The area of a triangle is exactly one half the area of the rectangle with the same base and height.

18.1 Turning integrals into the area of a rectangle

- How do we turn area under a curve like x^3 into a rectangle?

Intuition

- For $y = f(x) \in C^1$,
$$dy = f'(x)\ dx.$$
- So how does that work with integrals? Remember goal is to find
$$\int_{x=0}^{2} x^3\ dx.$$
  - First, find y such that for $y = f(x)$, $dy = f'(x)\ dx$. So want $f(x)$ such that $f'(x) = x^3$. Since $[x^4/4]' = (4/4)x^3 = x^3$,
$$1\ dy = x^3\ dx \Rightarrow \int_{x=0}^{2} x^3\ dx = \int_{x=0}^{2} 1\ dy.$$
  - Second, change the limits from x to y. When $x = 0$, $y = 0^4/4 = 0$. When $x = 2$, $y = 2^4/4 = 4$. So
$$\int_{x=0}^{2} x^3\ dx = \int_{y=0}^{4} 1\ dy.$$
  - Third, find the area of the simple integral in y!
$$\int_{y=0}^{4} dy = 1(4-0) = 4$$

Generalize

- Want $\int_{x=a}^{b} f(x)\ dx$.
- Suppose $w = F(x)$, and $w' = F'(x) = f(x)$. Then w runs from $F(a)$ to $F(b)$
- Also $dw/dx = f(x)$, so $f(x)\ dx = dw$. So

$$\int_{x=a}^{b} f(x)\ dx = \int_{w=F(a)}^{F(b)} dw = F(b) - F(a).$$

Theorem 5 (Fundamental Theorem of Calculus)
Let $f \in C^0[a,b]$ and F be any function with $F'(x) = f(x)$. Then

$$\int_{a}^{b} f(x)\ dx = F(b) - F(a).$$

18.2 Using the FTC in practice

- To solve $\int_a^b f(x)\ dx$
- Find a function $F(x)$ whose derivative is $F'(x) = f(x)$
- Calculate $F(b) - F(a)$

Definition 46
Suppose $F'(x) = f(x)$. Say that F is an **antiderivative** of f, write $F(x) \in \text{antider}_x(f(x))$ (or $\text{ad}_x(f(x))$ for short.)

- Often abuse notation and write $n^2 = O(n^3)$ instead of $n^2 \in O(n^3)$.
- Same with antiderivatives, write
$$\frac{x^3}{3} = \text{ad}_x(x^2),$$
instead of $(x^3/3) \in \text{ad}_x(x^2)$.
- Note $x^3/3$ is an antiderivative of x^2, since $[x^3/3]' = x^2$
- $x^3/3 + 4$ is also an antiderivative of x^2, since $[x^3/3 + 4]' = [x^3/3]' + [4]' = x^2$.

Lemma 29
If F is an antiderivative of f, so is $F + C$ for any real constant C. Conversely, if F and G are antiderivatives of f, then $F - G = C$ for some constant C.

Notes

- Sometimes write $\text{ad}_x(x^2) = x^3/3 + C$ to indicate that the family of antiderivatives is $x^3/3$ plus any constant.
- Don't need C for FTC, any antiderivative F works.
- Do need it for solving differential equations as we'll see later in the course.
- Nice property: antiderivatives are linear operators

Lemma 30
Suppose $F = \text{ad}_x(f(x))$ and $G = \text{ad}_x(g(x))$. Then for any $a, b \in \mathbb{R}$, $aF + bG = \text{ad}_x(af(x) + bg(x))$.

Example: stopping car

- A car traveling at 55 mph takes 6 seconds to stop. Assuming the brakes deliver constant acceleration, how many feet did it travel in this time?
- First step: need to figure out velocity

$$55\frac{\text{miles}}{\text{hour}} = 55\frac{5280 \text{ feet}}{3600 \text{ seconds}} = 80\ 2/3 \text{ ft/s}$$

  Acceleration is the derivative of velocity, so saying the brake deliver constant acceleration means the velocity is a line.

  So $v(0) = 80\ 2/3$, $v(6) = 0$, the line that goes through these two points is about

$$v(t) = 80\ 2/3 - [(80\ 2/3)/6]t.$$

- So our goal is to find

$$\int_{t=0}^{6} 242/3 - t \cdot 242/18\ dt.$$

- An antiderivative of 242/3 is $(242/3)t$. An antiderivative of $t(242/18) = (t^2/2)(242/18)$. So

$$\begin{aligned}\int_{t=0}^{6} 242/3 - t \cdot 242/18\ dt &= (242/3)(6-0) - (242/18)(6^2/2 - 0^2/2) \\ &= 242[2-1] = \boxed{242 \text{ feet}}\end{aligned}$$

Example: pouring concrete

- Concrete is poured into a mold at rate $5t/(t+5)$ cubic feet per sec. If the mold contains 1000 cubic feet, how many seconds does it take to fill?
- The problem: find T such that

$$\int_{t=0}^{T} \frac{5t}{t+5}\ dt = 1000.$$

- Helps to use algebra trick:

$$\begin{aligned}\frac{t}{t+5} &= 1 + g(t) \\ g(t) &= \frac{t}{t+5} - 1 \\ &= \frac{t}{t+5} - \frac{t+5}{t+5} \\ &= \frac{-5}{t+5}\end{aligned}$$

  so

$$\frac{t}{t+5} = 1 - \frac{5}{t+5}$$

$$\int_{t=0}^{T} \frac{5t}{t+5}\ dt = 5\int_{t=0}^{T} 1 - \frac{5}{t+5}\ dt.$$

  The antiderivative of 1 is t, the antiderivative of $1/(t+5)$ is $\ln(t+5)$. So

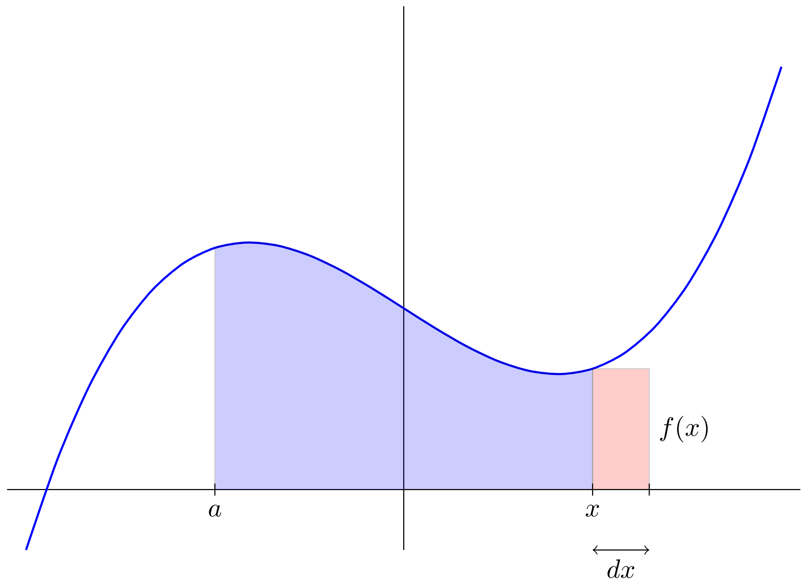
$$\int_{t=0}^{T} \frac{5t}{t+5}\ dt = 5[t - 5\ln(t+5)]|_0^T = 5[T - 5\ln(T+5) + 5\ln(5)]$$

- Setting this equal to 1000 gives a transcendental equation that can be solved numerically using bisection method:

$$\boxed{t = 219.01129276\ldots \approx 219.0 \text{ seconds}}$$

18.3 Second form of FTC

Another way to look at the FTC: suppose the upper limit is the variable. How does the area change when the upper limit changes?



Idea: for f continuous, let

$$A(x) = \int_a^x f(x)\ dx.$$

Then

$$A(x+h) - A(x) = \int_x^{x+h} f(x)\ dx = f(x)h + o(|h|).$$

Dividing by h and taking the limit as $h \to 0$ gives:

Theorem 6 (Second form of Fundamental Theorem of Calculus)
For f continuous over $[a, b]$,

$$\frac{d}{dx}\int_a^x f(s)\ ds = f(x).$$

19 Antiderivatives

Question of the Day What is the antiderivative of e^{3x}?

Today

- General antiderivatives
- Six antiderivatives to memorize

19.1 Every continuous function has an antiderivative

Recall the second form of the Fundamental Theorem of Calculus, for f continuous

$$\left[\int_{s=a}^{x} f(s)\ ds\right]' = f(x).$$

Fact 17 (Continuous functions have antiderivatives)
Every continuous function over $[a,b]$ has an antiderivative over $[a,b]$,

$$(\forall a \in \mathbb{R})\left(\int_a^x f(s)\ ds \in \mathrm{ad}_x(f(x))\right).$$

Proof. Let $f \in C^0_{[a,b]}$. Then let

$$g(x) = \int_{s=a}^{x} f(s)\ ds.$$

By the FTC, $g'(x) = f(x)$. □

Recall

- There were six derivatives I said just to memorize:

$f(x)$	$f'(x)$
x^a	ax^{a-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\exp(x)$	$\exp(x)$
$\arctan(x)$	$1/(1+x^2)$
$\ln(x)$	$1/x$

- Each of these has a corresponding antiderivative rule

$f(x)$	$\mathrm{ad}_x(f(x))$
$x^a\ (a \neq -1)$	$x^{a+1}/(a+1)$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\exp(x)$	$\exp(x)$
$1/(1+x^2)$	$\arctan(x)$
$1/x = x^{-1}$	$\ln(x)$

Mnemonics for the Power Rule

- So $[x^a]' = ax^{a-1}$ and $\mathrm{ad}_x(x^a) = x^{a+1}/(a+1)$ (for $a \neq 1$)
- Sometimes hard to remember which is which
- First mnemonic: division
$$\frac{x^a}{x} = x^{a-1}, \quad \frac{dx^a}{dx} = ax^{a-1}$$
- Second mnemonic: triangle
$$\int_0^1 x\ dx = 1/2 \Rightarrow \mathrm{ad}_x(x) = x^2/2$$

Mnemonic for $\sin$ **and** $\cos$

Again can be hard to remember where the negative sign goes

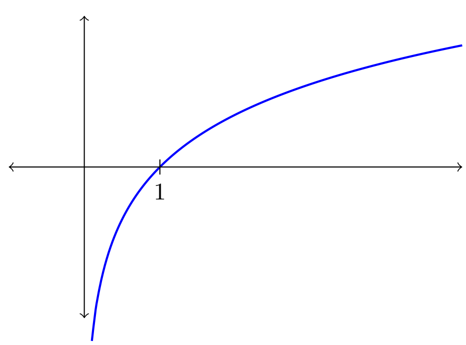
- Draw the unit circle
- As angle increases, move counter-clockwise, sin goes up, cos goes down

Mnemonic for e^x

Since $[e^x]' = \mathrm{ad}_x(e^x) = e^x$, no one ever has problems with this one

Mnemonics for $\ln(x)$

- $\ln(x)$ is the inverse of e^x, $[\ln(x)]'$ is the multiplicative inverse $1/x$
- Graph of $\ln(x)$:



- What is $\mathrm{ad}_x(1/x) = \mathrm{ad}_x(x^{-1})$? The power rule does not apply since you cannot divide by $-1+1$. One antiderivative is $g(x) = \mathrm{ad}_x(1/x) = \int_{s=1}^{x} 1/x\ dx$.
  - $g(1) = \int_1^1 1/x\ dx = 0$
  - $g(x)$ is an increasing function
  - $\int_1^x 1/x \,\mathrm{d}\,x \leq \int_1^x 1\ dx = x - 1$
  - So I want an elementary function that is 1) 0 at 1, 2) is increasing, 3) is smaller than $x - 1$.
  - $\ln(x)$ fits the bill!

19.2 Antiderivatives of functions of lines

- A line is $f(x) = ax + b$ for constants a and b
- Consider $[\exp(3x - 4)]'$
- Use the chain rule:
$$[\exp(3x-4)]' = \exp(3x-4)[3x-4]' = 3\exp(3x-4)$$

Qotd

- Want $\mathrm{ad}_x(e^{3x})$.
- By chain rule $[e^{3x}]' = 3e^{3x}$
- Divide both sides of equation by 3 to get

$$\left[\frac{e^{3x}}{3}\right]' = e^{3x},$$

  which means

$$\mathrm{ad}_x(e^{3x}) = e^{3x}/3.$$

More generally...

- To find antiderivative of exp,sin,cos,power of $ax + b$...
- Use regular antiderivative, then divide by a.

Examples

- Find an antiderivative of $1/(2x + 5)$
- A: $\ln(2x + 5)/2$
- Find an antiderivative of $\exp(-x)$
- A: $\exp(-x)/(-1) = -\exp(-x)$
- What is $\mathrm{ad}_x(\cos(\pi x))$?
- A: $\sin(\pi x)/\pi$

19.3 Notation for antiderivatives

Another common notation for antiderivatives is

$$\mathrm{ad}_x(f(x)) = \int f(x)\ dx.$$

Some peoples call an antiderivative an *indefinite integral*

I do not use this notation for the following reasons

- The antiderivative of a function (like the derivative) is another function. The integral of a function is a number that depends on the limits of integration. An antiderivative is not any type of integral, and calling it one leads to confusion over what an integral is.
- The notation is overloaded (it means too different things)

$$\int f(x)\ dx = \int_{x \in \mathbb{R}} f(x)\ dx.$$

20 Integration by parts

Question of the Day What is $\int_0^\pi x\sin(x)\ dx$?

Today

- Integration by parts

Integration by Parts

- IBP is the antidifferentiation counterpart to the product rule
- Recall for two functions f and g of x:

$$\frac{d}{dx}[fg] = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Writing using the prime notation:

$$[fg]' = f'g + g'f.$$

Rearrange this equation:

$$fg' = [fg]' - f'g.$$

Antidifferentiating gives you integration by parts:

$$\text{ad}_x(fg') = fg - \text{ad}_x(f'g).$$

Definition 47
The **integration by parts technique** (or IBP) writes the function to be antidfferentiated as fg', then applies the formula to get:

$$\text{ad}_x(fg') = fg - \text{ad}_x(f'g).$$

When to use Integration by Parts

- Gets used a lot in more advanced courses
- For us, three main types of problems

1: $x^d g'(x)$, where d is a positive integer and $g'(x)$ is an easy function to antidifferentiate like exp, square root, sin, or cos.

2: $x^c\ln(x)$, where c is any real number. When $c = 0$, this gives an antiderivative of $\ln(x)$.

3: $e^x\cos(x)$ or $e^x\sin(x)$.

- Note: in any of the above cases, you can replace one or more of the x with a linear function of x and IBP still works.
- For instance,

$$(x+3)^4\ln(x),\ e^{-x}\cos(\pi x),\ x^3\sin(x-\pi),$$

can all be found using IBP.

20.1 $h(x)$ is x raised to a positive integer times something we can antidifferentiate.

Example:

$$\text{Find } \operatorname{ad}_x(x\sin(x)).$$

This is x^1 times $\sin(x)$, which we know how to antidifferentiate.
In these types of problems, always set f equal to the x^{power} part
And g' equal to the thing we know how to antidifferentiate:

$$\begin{aligned} f(x) &= x \\ g'(x) &= \sin(x) \end{aligned}$$

Then figure out f' and g:

$$\begin{aligned} f'(x) &= 1 \\ g(x) &= -\cos(x) \end{aligned}$$

[Note you can let g be *any* antiderivative of g'.]
Now plug in the formula:

$$\begin{aligned} \operatorname{ad}_x(x\sin(x)) &= \operatorname{ad}_x(fg') \\ &= fg - \operatorname{ad}_x(f'g) \\ &= x(-\cos(x)) - \operatorname{ad}_x(-\cos(x)) \\ &= -x\cos(x) + \operatorname{ad}_x(\cos(x)) \\ &= \boxed{-x\cos(x) + \sin(x)}. \end{aligned}$$

- Start with simple example: $\operatorname{ad}_x(x\cos(x))$.
- Suppose I try $x\sin(x)$.
- Then use product rule:

$$[x\sin(x)]' = x[\sin(x)]' + [x]'\sin(x) = x\cos(x) + \sin(x).$$

- So rearranging,

$$x\cos(x) = [x\sin(x)]' - \sin(x),$$

- Taking antiderivative

$$\operatorname{ad}_x(x\cos(x) = x\sin(x) - \operatorname{ad}_x(\sin(x)) = x\sin(x) + \cos(x).$$

- Works on functions of the form:

$$x^{\text{some integer power}} \cdot (\text{something we can antidifferentiate}).$$

Fact 18 (Repeated IBP)
To antidifferentiate $x^d g'(x)$ where d is a positive integer and $g'(x) \in \{\cos(x), \sin(x), \exp(x), x^c\}$ requires d uses of IBP.

Example:

$$\text{Find } \text{ad}_x(x^2\exp(-2x)).$$

- Since $d = 2$, have to use IBP twice
- First use:

$$\begin{aligned} f_1(x) &= x^2 & f_1'(x) &= 2x \\ g_1'(x) &= \exp(-2x) & g_1(x) &= \exp(-2x)/(-2). \end{aligned}$$

  gives

$$\begin{aligned} \text{ad}_x(x^2\exp(-2x)) &= x^2\exp(-2x)/(-2) - \text{ad}_x(2x\exp(-2x)/(-2)) \\ &= -(1/2)x^2\exp(-2x) + \text{ad}_x(x\exp(-2x)). \end{aligned}$$

- Now use IBP again to find $\text{ad}_x(x\exp(-2x))$:

$$\begin{aligned} f_2(x) &= x & f_2'(x) &= 1 \\ g_2'(x) &= \exp(-2x) & g_2(x) &= \exp(-2x)/(-2). \end{aligned}$$

  gives

$$\begin{aligned} \text{ad}_x(x\exp(-2x)) &= x\exp(-2x)/(-2) - \text{ad}_x(\exp(-2x)/-2) \\ &= -(1/2)x\exp(-2x) + (1/2)\exp(-2x)/(-2) \\ &= -(1/2)x\exp(-2x) - (1/4)\exp(-2x). \end{aligned}$$

  Plugging back into above gives:

$$\begin{aligned} \text{ad}_x(x^2\exp(-2x)) &= -(1/2)x^2\exp(-2x) - (1/2)x\exp(-2x) - (1/4)\exp(-2x) \\ &= -(1/4)\exp(-2x)[2x^2 + 2x + 1]. \end{aligned}$$

20.2 $h(x)$ is x raised to any real power times $\ln$

Example

$$\text{Find } \text{ad}_x(\sqrt{x}\ln(x)).$$

- For these types of problems, always let $f(x) = \ln(x)$, and $g'(x) = x^c$.

$$\begin{aligned} f(x) &= \ln(x) & f'(x) &= 1/x \\ g'(x) &= x^{1/2} & g(x) &= x^{3/2}/(3/2) \end{aligned}$$

  gives

$$\begin{aligned} \text{ad}_x(x^{1/2}\ln(x)) &= (2/3)x^{3/2}\ln(x) - \text{ad}_x((2/3)x^{3/2}/x) \\ &= (2/3)x^{3/2}\ln(x) - (2/3)\,\text{ad}_x(x^{1/2}) \\ &= (2/3)x^{3/2}\ln(x) - (2/3)x^{3/2}/(2/3) \\ &= x^{3/2}[(2/3)\ln(x) - (4/9)] \end{aligned}$$

20.3 $h(x)$ is $\exp$ times $\cos$ (or $\sin$)

- A tricky place to use integration by parts is on problems of form:

$$\text{ad}_x(\exp(c_1x + c_2)\sin(x)) \text{ or } \text{ad}_x(\exp(c_1x + c_2)\cos(x)).$$

- Can use integration by parts twice to get the answer.
- Example: find $\mathrm{ad}_x(e^{2x}\sin(x))$.
- Let $F(x) = \mathrm{ad}_x(\exp(2x)\sin(x))$
- Let $f_1(x) = \exp(2x)$, $g_1'(x) = \sin(x)$
- Then $f_1'(x) = 2\exp(2x)$, $g_1(x) = -\cos(x)$

$$\begin{aligned} F(x) &= -\exp(2x)\cos(x) - \mathrm{ad}_x\,(-\cos(x))2\exp(2x) \\ &= -\exp(2x)\cos(x) + 2\,\mathrm{ad}_x\exp(2x)\cos(x). \end{aligned}$$

- Now use IBP again on the last part.
- $f_2(x) = \exp(2x)$, $g_2'(x) = \cos(x)$
- $f_2'(x) = 2\exp(2x)$, $g_2(x) = \sin(x)$

$$\begin{aligned} \mathrm{ad}_x\exp(2x)\cos(x) &= \exp(2x)\sin(x) - \mathrm{ad}_x(2\exp(2x)\sin(x)) \\ &= \exp(2x)\sin(x) - 2F(x). \end{aligned}$$

- Putting that in for earlier gives:

$$\begin{aligned} F(x) &= -\exp(2x)\cos(x) + 2exp(2x)\sin(x) - 4F(x) \\ 5F(x) &= -\exp(2x)\cos(x) + 2\exp(2x)\sin(x) \\ F(x) &= \exp(2x)[(2/5)\sin(x) - (1/5)\cos(x)]. \end{aligned}$$

21 Substitution/Change of Variables

Question of the Day Find $\int_0^{\sqrt{\pi}} x\sin(x^2)\ dx$.

Today

- Substitution aka change of variables

Substitution

- Used whenever you have a function of a nonlinear function.
- Examples:
$$\exp(x^2),\quad \sin(\cos(x)),\quad \ln(\sin(x))\cos(x).$$

21.1 Substitution for Integrals (LID)

- Recall our intuition behind the Fundamental Theorem of Calculus:
$$dr = f'(s)\ ds.$$
This allows us to change the differential from ds to dr:
$$ds = \frac{1}{f'(s)}\ dr.$$
- If $s \in [a, b]$, then the limits change to $f(a)$ to $f(b)$.
- Finally, write the integrand in terms of r instead of s.

Limits, Integrand, Differential (LID) Limits, Integrand, and differential are the three pieces of an integral.

1: Change the limits.

2: Change the differential.

3: Change the integrand.

How do you choose $f(s)$?

- Pick $r = f(s)$ to make problem more simple.
- In QotD, $\sin(x^2)$ is hard. By making $t = x^2$, now it is $\sin(t)$, which is simpler. Let's go through our steps for $t = x^2$:

  1: Limits: $x \in [0, \sqrt{\pi}] \to t \in [0^2, \pi]$

  2: Differential: $\frac{dt}{dx} = 2x \Rightarrow dx = \frac{1}{2x}\ dt$. At this point the transformation is
$$\int_{x=0}^{\sqrt{\pi}} x\sin(x^2)\ dx = \int_{t=0}^{\pi} x\sin(x^2)\frac{1}{2x}\ dt$$

  3: Integrand: replace any x^2 with t to get:
$$\int_{x=0}^{\sqrt{\pi}} x\sin(x^2)\ dx = \int_{t=0}^{\pi} (1/2)\sin(t)\ dt$$

  Now use $\text{antider}_t(\sin(t)) = -\cos(t)$ to get:
$$\begin{aligned}\int_{t=0}^{\pi} (1/2)\sin(t)\ dt &= -(1/2)\cos(t)|_0^{\pi} \\ &= -(1/2)(-1) - (-(1/2)(1)) = \boxed{1}.\end{aligned}$$

21.2 Substitution for Antiderivatives

- Start with the chain rule:
$$[f(g(x))]' = f'(g(x))g'(x).$$
- So when you take the derivative of a function of a function, you multiply by the derivative of the inner function.
- So when you antidifferentiate, you divide by the derivative of the inner function.

Definition 48
The **substitution technique** (also known as **change of variables** can be written

$$\operatorname{ad}_x(f(x)) = \operatorname{ad}_w\left(\frac{f(x)}{w'(x)}\right).$$

Substitution steps

1: Look for an inner nonlinear function inside an outer function

2: Set w equal to inner function

3: Use formula: $\operatorname{ad}_x(f(x)) = \operatorname{ad}_w(f(x)/w'(x))$.

4: Write $f(x)/w'(x)$ in terms of w.

5: Solve new problem.

6: Put back x variable for w.

Example

- Find $\operatorname{ad}_x(x^3 e^{-x^2})$

1: Note that $-x^2$ is inside exp

2: So let $w = -x^2$

3: Hence $w' = -2x$, and

$$\operatorname{ad}_x(x^3 e^{-x^2}) = \operatorname{ad}_w(x^3 e^{-x^2}/(-2x)) = \operatorname{ad}_w((-1/2)x^2 e^{-x^2}).$$

4: Putting in $w = -x^2$:

$$\operatorname{ad}_w(-(1/2)x^2 e^{-x^2}) = \operatorname{ad}_w((1/2)we^w) = (1/2)\operatorname{ad}_w(we^w).$$

5: Solving this problem requires integration by parts:

$$\begin{aligned} f &= w & f' &= 1 \\ g' &= e^w & g &= e^w \end{aligned}$$

So

$$\operatorname{ad}_w(we^w) = we^w - \operatorname{ad}_w(e^w) = we^w - e^w.$$

and

$$(1/2)\operatorname{ad}_w(we^w) = (1/2)e^w(w-1)$$

6: Hence $\operatorname{ad}_x(x^3 e^{-x^2}) = -(1/2)e^{-x^2}(x^2+1)$.

Checking answers

Can always check out answers by differentiating:

$$\begin{aligned}[-(1/2)\cos(x^2)]' &= -(1/2)[\cos(x^2)]' \\ &= -(1/2)(-\sin(x^2))(2x) = x\sin(x^2)\end{aligned}$$

Similarly, from the product rule and chain rule:

$$\begin{aligned}[-(1/2)e^{-x^2}(x^2+1)]' &= -(1/2)[\exp(-x^2)(2x) + (-2x)\exp(-x^2)(x^2+1)] \\ &= -(1/2)\exp(-x^2)[2x - 2x^3 - 2x] \\ &= x^3\exp(-x^2).\end{aligned}$$

Example

$$\text{Find } \operatorname{ad}_r(1/(2+e^r)).$$

- Note the "nonlinear function in a function" can hide a little bit.
- Rewrite the function as $(2+e^r)^{-1}$.

1: $2+e^r$ is inside (stuff)$^{-1}$.

2: Inner function $w(r) = 2+e^r$.

3:

$$\operatorname{ad}_x((2+e^r)^{-1}) = \operatorname{ad}_w((2+e^r)^{-1}/e^r).$$

4: Write in terms of w:

$$\operatorname{ad}_w((2+e^r)^{-1}/e^r) = \operatorname{ad}_w([w(w-2)]^{-1}).$$

5: Later on, we'll see how to systematically tackle problems like this. For now, note

$$\frac{1}{w(w-2)} = \frac{(1/2)}{w-2} - \frac{1/2}{w}.$$

so

$$\operatorname{ad}_w\left(\frac{1}{w(w-2)}\right) = (1/2)\ln(w-2) - (1/2)\ln(w).$$

6: Put the x back in:

$$\operatorname{ad}_x(1/(2+e^r)) = (1/2)\ln(e^r) - (1/2)\ln(e^r+2) = (1/2)r - (1/2)\ln(2+e^r).$$

- Checking the answer:

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{2}r - \frac{1}{2}\ln(2+e^r)\right] &= \frac{1}{2} - \frac{e^r}{2(2+e^r)} = \frac{2+e^r-e^r}{2(2+e^r)} \\ &= \frac{2}{2(2+e^r)} = \frac{1}{2+e^r}\end{aligned}$$

Example

$$\text{Find } \operatorname{ad}_\theta(\sin(\theta)\cos^2(\theta)).$$

1: Have a $\cos(\theta)$ being squared.

2: Let $w = \cos(\theta)$.

3:

$$\operatorname{ad}_\theta(\sin(\theta)\cos^2(\theta)) = \operatorname{ad}_w(\sin(\theta)\cos^2(\theta)/(-\sin(\theta))).$$

4:
$$\text{ad}_w(\sin(\theta)\cos^2(\theta)/(-\sin(\theta))) = \text{ad}_w(-\cos^2(\theta)) = \text{ad}_w(-w^2).$$

5: $\text{ad}_w(-w^2) = -w^3/3$

6:
$$\text{ad}_\theta(\sin(\theta)\cos^2(\theta)) = -\cos^3(\theta)/3.$$

When to use substitution?

- Consider these examples:

$$f_1(x) = x^2 \exp(-x^3) \qquad f_2(x) = x^2 \exp(-3x+2)$$
$$f_3(x) = x\sqrt{1-x^2} \qquad f_4(x) = \sqrt{3+4x}.$$

- When should you use substitution?
- For f_1, $-x^3$ (a nonlinear function) is inside the exp function, so use substitution.
- For f_2, $-3x+2$ is a linear function, so there is no need for substitution (although it won't hurt to use $w = -3x+2$, it does take longer to solve the resulting integral.)
- For f_3, $\sqrt{1-x^2} = (1-x^2)^{-1/2}$. Definitely a nonlinear function inside $\sqrt{}$, so $w = 1-x^2$.
- For f_4, since what's inside the $\sqrt{}$ is a linear function, you don't need substitution (although it won't hurt if you use it).

22 Partial Fractions and rational functions

Question of the Day Find an antidervative of

$$\frac{1}{(x-2)(x-3)}$$

Today

- Partial fractions

22.1 Linear function in the denominator

Example

- What is $\int_{x=0}^{1} 1/(2x+3)\ dx$?
- $\text{antider}_s(1/s) = \ln(s)$, $\text{antider}_x(1/(2x+3)) = \ln(2x+3)/2$
- Answer: $(1/2)[\ln(6) - \ln(3)] = \ln(2)/2 \approx 0.3465$

Example

- Find an antiderivative of $1/(2x+3)^4$
- Use the power rule with power -4:

$$\text{ad}_x((2x+3)^{-4}) = \frac{(2x+3)^{-3}}{-3} \cdot \frac{1}{2} = -\frac{1}{6(2x+3)}$$

22.2 Product of linear functions in the denominator

The idea

- First an example using numbers

$$\frac{1}{12} = \frac{1}{3 \cdot 4} = \frac{1}{3} - \frac{1}{4}$$

- Can also use polynomials

$$\begin{aligned}\frac{1}{x-3} - \frac{1}{x-2} &= \frac{x-2}{(x-2)(x-3)} - \frac{x-3}{(x-2)(x-3)} \\ &= \frac{1}{(x-2)(x-3)}\end{aligned}$$

  So that means

$$\begin{aligned}\text{ad}_x\left(\frac{1}{(x-2)(x-3)}\right) &= \text{ad}_x\left(\frac{1}{x-3} - \frac{1}{x-2}\right) \\ &= \boxed{\ln(x-3) - \ln(x-2)}\end{aligned}$$

- In fact, I can do more! For any a and b, there exist c_1 and c_2 such that

$$\frac{ax+b}{(x-2)(x-3)} = \frac{c_1}{x-3} + \frac{c_2}{x-2}.$$

- In the qotd, $a = 0$, $b = 1$, $c_1 = 1$ and $c_2 = -1$.

Partial fractions method Breaking a fraction into partial fractions

- Let $L_1(x)$ and $L_2(x)$ be lines
- To write
$$\frac{ax+b}{L_1(x)L_2(x)} = \frac{c_1}{L_1(x)} + \frac{c_2}{L_2(x)}$$
first multiply both sides of the equation by $L_1(x)L_2(x)$ to get
$$ax + b = c_1L_2(x) + c_2L_1(x).$$
- Next, plug in the values r_1 and r_2 where $L_1(r_1) = 0$ and $L_2(r_2) = 0$ to get c_1 and c_2.

Notes

- I used r_1 and r_2 for where $L_1(x) = 0$ and $L_2(x) = 0$ because the inputs where a function value is zero is called a *root* of the function. Also called a *zero* of the function.

Example

- Find the partial fraction representation of
$$\frac{3x-4}{(2x-6)(x+2)}.$$
- Here $L_1(x) = 2x - 6$. $2x - 6 = 0 \Rightarrow r_1 = 3$
- $L_2(x) = x + 2$. $x + 2 = 0 \Rightarrow r_2 = -2$
- Now take the representation:
$$\frac{3x-4}{(2x-6)(x+2)} = \frac{c_1}{2x-6} + \frac{c_2}{x+2}$$
multiply through by $(2x - 6)(x + 2)$:
$$3x - 4 = c_1(2x - 6) + c_2(x + 2)$$
Put in $x = 3$ to get
$$5 = 7c_2 \Rightarrow c_2 = 5/7$$
Put in $x = -2$ to get
$$-10 = c_1(-10) \Rightarrow c_1 = 1$$
So
$$\frac{3x-4}{(2x-6)(x+2)} = \frac{1}{2x-6} - \frac{5/7}{x+2}.$$

22.3 Quadratics in the denominator

A quadratic has from $ax^2 + bx + c$

- Recall $\mathrm{ad}_x(1/(x^2 + 1)) = \arctan(x)$
- Find $\mathrm{ad}_x(1/(3x^2 + 1))$
- $$\mathrm{ad}_x\left(\frac{1}{3x^2+1}\right) = \mathrm{ad}_x\left(\frac{1}{(\sqrt{3}x)^2+1}\right) = \frac{\arctan(\sqrt{3}x)}{\sqrt{3}}.$$

General rational functions

Definition 49
A **polynomial** is a function of the form $f(x) = 0$ or

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

where $a_n \neq 0$. If $f(x) = 0$, the **degree** of the polynomial is $\deg(f) = 0$. If $f(x) \neq 0$, the **degree** of the polynomial is n.

Example

- $\deg(x^2 + 3x - 2) = 2, \quad \deg(x^3 - 2) = 3, \quad \deg(4) = 0.$

Definition 50
A function which is a polynomial divided by another polynomial is called a **rational function**.

In turns out that by using advanced partial fraction techniques, all rational functions can be antidifferentiated using elementary functions.

Lemma 31
All rational functions have an antiderivative that is the sum of terms of the form $a\ln(bx+c)$, $a\arctan(bx+c)$, $a/(bx+c)^k$, or $(ax+b)/(cx+d)^k$.

23 Exponential decay and separable differential equations

Question of the Day A wood beam has 20% of the the original ^{14}C remaining. How old is it?

Today

- Exponential Decay
- Carbon dating

Many things grow or decay exponentially

- Populations (with unlimited resources)
- disease (early stage)
- Continuously compounded interest
- Radioactive materials

Let t = time and $s(t)$ = amount of substance at time t.

The Model

$$\underbrace{ds}_{\text{change in amount}} = \overbrace{s(t)}^{\text{Amount at time } t} \cdot \underbrace{k}_{\text{constant}} \cdot \overbrace{dt}^{\text{change in time}}$$

- Exponential growth: When $k > 0$, population is increasing
- Exponential decay: When $k < 0$, population is decreasing
- $ds = ks\ dt$ is an example of a *differential equation*
- Recall: $ds = f'(t)\ dt$ if $s = f(t)$

$$ds = sk\ dt \Leftrightarrow \frac{ds}{dt} = ks \Leftrightarrow \frac{1}{s}\ ds = k\ dt.$$

- Called separation of variables, since s and ds are on one side and t and dt are on the other.
- Now integrate both sides:

$$\begin{aligned} \int_{s=s(0)}^{s(T)} \frac{1}{s}\ ds &= \int_{t=0}^{T} k\ dt \\ \ln(s(T)) - \ln(s(0)) &= k(T-0) \\ \ln(s(T)/s(0)) &= kT \\ s(T)/s(0) &= e^{kT} \\ s(T) &= s(0)e^{kT}. \end{aligned}$$

- Put back dummy variable t for T to get:

$$\boxed{s(t) = s(0)\exp(kt)}$$

is the solution to $ds = ks\ dt$. This proves the following lemma.

Lemma 32 (Exponential growth/decay)
If $s' = ks$ (or $ds = ks\ dt$) then
$$s(t) = s(0)\exp(kt)$$
is the unique solution.

Definition 51
A **differential equation** or **DE** is an equation that involves one or more derivatives.

Definition 52
The **order** of a differential equation is number of the highest derivative in the equation.

Example

- For instance, $ds = ks\ dt$ which is the same as $s' = ks$ is a first order DE, since it only contains one derivative.
- The DE $y'' - x^2 y = e^x$ is a second order DE

Definition 53
The **separation of variables** technique for $y' = f(x)g(y)$ is
$$\int_{y=y(a)}^{y(b)} \frac{1}{g(y)}\ dy = \int_{x=a}^{b} f(x)\ dx.$$

Half life and k

Definition 54
For a radioactive substance, the **half life** is the value of t such that $s(t) = (1/2)s(0)$, so for $s(t) = s(0)e^{kt}$, the half life is $\ln(1/2)/k$.

- Radioactive materials decay rates usually given via half life.
- For instance, ^{14}C has a half life of about 5730 years.
- After 5730 years, 1/2 of original material remains.
- After $(2)(5730) = 11460$, $(1/2)(1/2) = 1/4 = 25\%$ of original material remains
- After $(3)(5730)$, $1/8 = 12.5\%$ remains
- So how long until 20% of original remains?
- Find k:
$$\begin{aligned}(1/2)s(0) = s(0)\exp(k(5730)) &\Rightarrow 1/2 = \exp(-5730k)\\ &\Rightarrow \ln(1/2) = 5730k\\ &\Rightarrow k = \ln(1/2)/5730 = -\ln(2)/5730.\end{aligned}$$
- Now turn equation around to find t:
$$\begin{aligned}(0.2)s(0) = s(0)\exp(t\ln(1/2)/5730) &\Rightarrow 0.2 = \exp(t\ln(1/2)/5730)\\ &\Rightarrow \ln(0.2) = t\ln(1/2)/5730\\ &\Rightarrow t = \frac{\ln(0.2)}{\ln(0.5)}5730\end{aligned}$$
$$t = (2.321\ldots)5730 \approx 13\,300 \text{ years.}$$
- Libby started in 1947, awarded Nobel Prize in Chemistry in 1960

Exponential growth

- Many things grow exponentially
- Viruses
- Interest compounded continuously

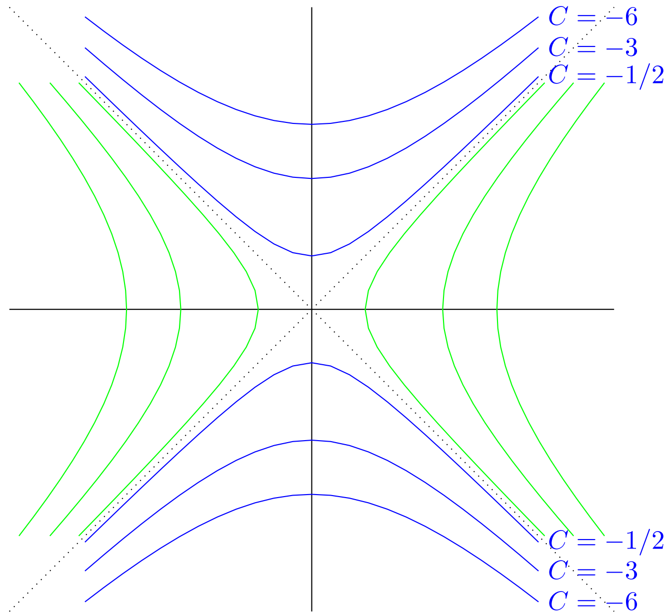
Example Interest is compounded continuously at rate 3% per year. How long does it take \$100 to double?

$$\begin{aligned} s &= s_0 \exp(kt) \\ 200 &= 100 \exp(0.03t) \\ 2 &= e^{0.03t} \\ \ln(2) &= 0.03t \\ t &= \ln(2)/0.03 \approx 23.10 \text{ years} \end{aligned}$$

The smaller k is (the interest rate), the longer it takes to double.

Separable D.E.'s

- Could get t and s on separate sides of equation
- General form:
$$g(t)\ ds = f(s)\ dt \Leftrightarrow \frac{1}{f(s)}\ ds = \frac{1}{g(t)}\ dt.$$
$$\text{antider}_s(f(s)^{-1}) + C = \text{antider}_t(g(t)^{-1}).$$
- Example:
$$\frac{dy}{dx} = \frac{x}{y}.$$
  - Step 1: Separate x and y:
$$y\ dy = x\ dx.$$
  - Step 2: Antidifferentiate both sides (with constant):
$$\frac{y^2}{2} + C_y = \frac{x^2}{2}.$$
  - Step 3: Sometimes possible to simplify:
$$C = x^2 - y^2$$
    [Note that $C = 2C_y$ is just a new constant.]
- These are hyperbolas



$C = -6$
$C = -3$
$C = -1/2$
$C = -1/2$
$C = -3$
$C = -6$

24 Improper Integrals

Question of the Day What is
$$\int_{x=1}^{\infty} 1/x^2 \ dx?$$

Today

- Improper integrals

Our story so far

- Defined Riemann integral for $\int_{x \in [a,b]} f(x) \ dx$.
- $[a,b] = \{x : a \leq x \leq b\}$
- Let $x_1, x_2, x_3, \ldots$ be any sequence of points in $[a,b]$ with a limit.
- Nice fact: $\lim_{n \to \infty} x_i \in [a,b]$.
- The interval $[a,b]$ is *closed* under the operation of taking limits.
- Now consider $(0,1) = \{x : 0 < x < 1\}$.
- Then $x_i = 1 - 1/i$ give points in $(0,1)$. But
$$\lim_{i \to \infty} x_i = 1 \notin (0,1).$$
- So $(0,1)$ is not *closed*.
- Now consider $[1,\infty)$. For any sequence of points in $[1,\infty)$ with a limit, the limit is in $[1,\infty)$. So $[1,\infty)$ is closed. However, it is not bounded.

Definition 55
An integral of the form
$$\int_{x \in A} f(x) \ dx,$$
where A is not compact, is called an **improper integral.**

24.1 Noncompact A

Turn noncompact sets into the union of compact sets.

Definition 56
Suppose
$$A = [a_1,b] \cup [a_2,b] \cup [a_3,b] \cup \cdots$$
or
$$A = [a,b_1] \cup [a,b_2] \cup [a,b_3] \cup \cdots$$
Let A_i stand for $[a_i,b]$ or $[a,b_i]$ as appropriate. Then
$$\int_A f(x) \ dx = \lim_{i \to \infty} \int_{A_i} f(x) \ dx.$$

In practice, this means to find the value of improper integrals, we use limits.

Qotd

- Note $[1,\infty) = [1,2] \cup [1,3] \cup [1,4] \cup \cdots$.
- Also, $1/x^2 \geq 0$ for all $x \in [1,\infty)$. So

$$\begin{aligned}\int_{x\in[1,\infty)} 1/x^2\ dx &= \lim_{b\to\infty} \int_{x\in[1,b]} x^{-2}\ dx \\ &= \lim_{b\to\infty} x^{-1}/(-1)|_1^b \\ &= \lim_{b\to\infty} 1 - 1/b \\ &= 1.\end{aligned}$$

Example:

$$\text{Find } \int_0^1 1/\sqrt{x}\ dx.$$

Since $1/\sqrt{x}$ is not defined at 0, this integral is really:

$$\int_{x\in(0,1]} 1/\sqrt{x}\ dx$$

so is improper! In this case,

$$(0,1] = [1/2,1] \cup [1/3,1] \cup [1/4,1] \cup \cdots$$

That means:

$$\begin{aligned}\int_0^1 1/\sqrt{x}\ dx &= \lim_{i\to\infty} \int_{x\in[1/i,1]} x^{-1/2}\ dx \\ &= \lim_{i\to\infty} x^{1/2}/(1/2)|_{1/i}^1 \\ &= \lim_{i\to\infty} 2 - 2\sqrt{1/i} \\ &= 2.\end{aligned}$$

Definition 57
A limit which exists and is finite is called *convergent* (say the function or sequence *converges.* If the limit does not exist, or is $+\infty$ or is $-\infty$, the limit is *divergent* (say the function or sequence *diverges.*

Definition 58
An improper integral where the limit does not exist, or is ∞ or $-\infty$, is called **divergent**. When the limit does exist, the integral is **convergent**.

Example

- Find $\int_0^1 1/x \,\mathrm{d}\, x$.
- Again $(0,1] = [1/2,1] \cup [1/3,1] \cup \cdots$, so

$$\begin{aligned}\int_0^1 1/x \,\mathrm{d}\, x &= \lim_{i\to\infty} \int_{1/i}^1 1/x\ dx \\ &= \lim_{i\to\infty} [\ln(1) - \ln(1/i)] \qquad = \infty\end{aligned}$$

- So this improper integral $\boxed{\text{diverges}}$ or we say is $\boxed{\text{divergent}}$.

Be careful when looking at if an integral is improper!

- Example:

$$\text{Is } \int_0^3 \frac{1}{x-1}\, dx \text{ improper?}$$

Yes! Since $1/(x-1)$ is not defined at 1, integral is really:

$$\int_{x\in[0,1)\cup(1,3]} \frac{1}{x-1}\, dx = \int_{x\in[0,1)} \frac{1}{x-1}\, dx + \int_{x\in(1,3]} \frac{1}{x-1}\, dx.$$

- Both integrals on RHS are improper.
- The integrand on one is negative. So what do we do?

24.2 Unbounded on both sides

- Note that the A_i have to be fixed on one side, either $[a_i, b]$ for all i or of the form $[a, b_i]$ for all i.
- What if they are not?
- Break A into two pieces.
  - Example, $A = (-\infty, \infty)$,

$$\int_A f(x)\, dx = \int_{x\in(-\infty,0]} f(x)\, dx + \int_{x\in[0,\infty)} f(x)\, dx.$$

    Did not have to break at 0, for instance,

$$\int_A f(x)\, dx = \int_{x\in(-\infty,1.5]} f(x)\, dx + \int_{x\in[1.5,\infty)} f(x)\, dx.$$

    Also works for intervals open on both ends:

$$\int_{x\in(0,1)} \frac{1}{\sqrt{x(1-x)}}\, dx = \int_{x\in(0,1/2]} \frac{1}{\sqrt{x(1-x)}}\, dx + \int_{x\in[1/2,1)} \frac{1}{\sqrt{x(1-x)}}\, dx.$$

- After the break up, need to know how to add and subtract ∞, $-\infty$

Definition 59
(Adding with ∞ in extended reals.) For $c \in \mathbb{R}$, addition and substraction with ∞ is defined as follows:

$$\infty - c = \infty$$
$$c - \infty = -\infty$$
$$\infty - \infty = \text{does not exist}$$

Example (continued) What is

$$\int_0^3 \frac{1}{x-1}\, dx?$$

Since it is negative for $x < 1$, this integral can be broken up as:

$$\int_0^3 \frac{1}{x-1}\, dx = \int_{x\in(1,3]} \frac{1}{x-1}\, dx - \int_{x\in[0,1)} \frac{-1}{x-1}\, dx$$

Working as before, both of these integrals turn out to be infinity, so the original integral $\boxed{\text{does not exist.}}$

Example (In probability, this integral arises in finding the normalizing constant of a normal distribution.) Find

$$I = \int_{x \in \mathbb{R}} x \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\ dx.$$

The integrand is positive when $x \geq 0$ and negative when $x \leq 0$. Hence

$$\begin{aligned} I &= \int_0^\infty x \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\ dx - \int_{-\infty}^0 (-x) \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\ dx \\ &= \lim_{b \to \infty} \int_0^b x \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\ dx - \lim_{a \to -\infty} \int_a^0 (-x) \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\ dx \end{aligned}$$

Note that the same change of variables: $w = x^2/2$ works for both integrals. Here $dw = x\ dx$, so

$$\begin{aligned} I &= \lim_{b \to \infty} \int_0^{b^2/2} \frac{1}{\sqrt{2\pi}} \exp(-w)\ dw - \lim_{a \to -\infty} \int_{a^2/2}^0 \frac{-1}{\sqrt{2\pi}} \exp(-w)\ dw \\ &= \lim_{b \to \infty} \left. \frac{-\exp(-w)}{\sqrt{2\pi}} \right|_0^{b^2/2} - \lim_{a \to -\infty} \left. \frac{\exp(-w)}{\sqrt{2\pi}} \right|_{a^2/2}^0 \\ &= \lim_{b \to \infty} \frac{1 - \exp(-b^2/2)}{\sqrt{2\pi}} - \lim_{a \to -\infty} \frac{1 - \exp(a^2/2)}{\sqrt{2\pi}} \\ &= (1 - 0) - (1 - 0) = 0. \end{aligned}$$

Example (This problem arises in probability, where it is known as finding the normalizing constant of a Cauchy distribution.) Find

$$\int_{x \in \mathbb{R}} \frac{1}{1 + x^2}\ dx.$$

Break it up into two pieces

$$\begin{aligned} \int_{-\infty}^\infty \frac{1}{1 + x^2}\ dx &= \int_{-\infty}^0 \frac{1}{1 + x^2}\ dx + \int_0^\infty \frac{1}{1 + x^2}\ dx \\ &= \lim_{a \to -\infty} \arctan(a)|_a^0 + \lim_{b \to \infty} \arctan(b)|_0^b \\ &= \pi/2 - (-\pi/2) = \pi \approx \boxed{3.141} \end{aligned}$$

25 Series tests: Divergence, Integral, and p-series

Question of the Day Does

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

converge?

Today

- Divergence test
- Integral test
- p-series test

25.1 The Divergence Test

Recall

- Series is the limit of a partial sum:
$$\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i.$$
- Series where the limit exists are **convergent**, and when the limit does not exist are **divergent**
- If $\lim_{n\to\infty} a_i$ does not exist, or exists and is not 0, then the series must be divergent.

Lemma 33 (The Divergence Test)
For a series $\sum_{i=k}^{\infty} a_i$, the only way the series can converge is if $\lim_{i\to\infty} a_i = 0$.

- So $1 + 1 + 1 + \cdots$ does not converge
- So $\sum i/(i+1)$ does not converge
- Does $\sum i/(i^2+1)$ converge? The divergence test does not tell us!

Converge, diverge, fail

- A series test has at most three outcomes

  1: The test reports the series converges

  2: The test reports the series diverges

  3: The test fails.

- In the last case, the test gives **no information** about whether the series converges or not.
- The divergence test reports either divergence or fails, it never can be used to prove convergence!
- In the qotd: $\lim_{i\to\infty} 1/i^2 = 0$, so the divergence test fails. Need a better test!

25.2 Integral test

- A series is area of positive bars minus area of negative bars
- An integral is area under positive function minus area under negative function
- Idea: use integrals to see how bars behave

Lemma 34 (Integral Test)
Suppose $a_i = f(i)$ for $i \in \{1, 2, 3, \ldots\}$ where f is a positive, continuous decreasing function of x for $x \geq 1$. Then $\sum_{i=c}^{\infty} a_i$ and $\int_{x=c}^{\infty} f(x)\,dx$ either both converge or both diverge.

- When integral test applies, always returns "converges" or "diverges", never fails!
- Qotd: $1/x^2$ is a positive, continuous, decreasing function. So

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \text{ converges} \Leftrightarrow \int_{x=1}^{\infty} \frac{1}{x^2}\,dx \text{ converges.}$$

  And

$$\begin{aligned}
\int_{x=1}^{\infty} \frac{1}{x^2}\,dx &= \lim_{b\to\infty} \int_{x=1}^{b} \frac{1}{x^2}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b \\
&= \lim_{b\to\infty} -1/b - (-1/1) \\
&= 1.
\end{aligned}$$

  Hence $\sum_{i=1}^{\infty} 1/i^2$ converges.

Example Does

$$\sum_{i=1}^{n} \frac{1}{i^{3/2}}$$

converge?

Again, $f(x) = x^{-3/2}$ is continuous, positive, and decreasing, so check

$$\begin{aligned}
\int_{x=1}^{\infty} x^{-3/2}\,dx &= \lim_{b\to\infty} \int_{x=1}^{b} x^{-3/2}\,dx \\
&= \lim_{b\to\infty} x^{-1/2}/(-1/2)|_1^b \\
&= \lim_{b\to\infty} -2b^{-1/2} + 2 \\
&= 2.
\end{aligned}$$

Example Does

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

converge?

- We know it doesn't (Harmonic series), but let's try integral test:

- $f(x) = 1/x$ is continuous, positive, decreasing:

$$\begin{aligned}
\int_{x=1}^{\infty} 1/x\ dx &= \lim_{b\to\infty} \int_{x=1}^{b} 1/x\ dx \\
&= \lim_{b\to\infty} \ln(x)|_1^b \\
&= \lim_{b\to\infty} \ln(b) - \ln(1) \\
&= \infty.
\end{aligned}$$

So the Harmonic series diverges. There's a pattern here!

Definition 60
A ***p*-series** is (for $p > 0$)

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

The p-series with $p = 1$ is called the **Harmonic Series**.

Lemma 35 (p-series test)
A p-series ($\sum_{i=1}^{\infty} 1/i^p$) converges if and only if $p > 1$.

Proof. The $p = 1$ case is the Harmonic series, which diverges. Suppose $p > 1$. Then $f(x) = 1/x^p$ is positive, continuous, and decreasing, so

$$\begin{aligned}
\int_{x=1}^{\infty} &= \lim_{b\to\infty} \int_{x=1}^{b} x^{-p}\ dx \\
&= \lim_{b\to\infty} x^{-p+1}/(-p+1)|_1^b \\
&= \lim_{b\to\infty} (1-p)^{-1} b^{1-p} - (1-p)^{-1} \\
&= (p-1)^{-1}.
\end{aligned}$$

Hence the p-series converges in this case.

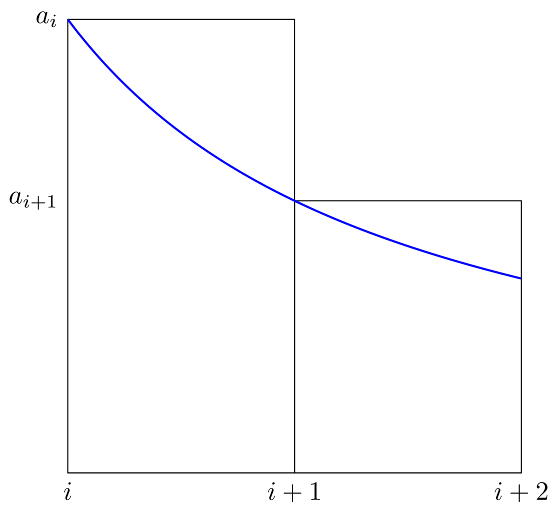
Now suppose $p \in (0, 1)$. Then

$$\begin{aligned}
\int_{x=1}^{\infty} &= \lim_{b\to\infty} \int_{x=1}^{b} x^{-p}\ dx \\
&= \lim_{b\to\infty} x^{-p+1}/(-p+1)|_1^b \\
&= \lim_{b\to\infty} (1-p)^{-1} b^{1-p} - (-(1-p)^{-1}) \\
&= \infty.
\end{aligned}$$

So in this case, the p-series diverges. □

Why does the integral test work?

- Start with a picture:



- Because $f(x)$ continuous,positive, decreasing

$$(\forall x \in [i, i+1])(a_{i+1} \le f(x) \le a_i).$$

- Integrating gives:

$$\int_{x=i}^{i+1} a_{i+1}\ dx \le \int_{x=i}^{i+1} f(x)\ dx \le \int_{x=i}^{i+1} a_i\ dx.$$

$$a_{i+1} \le \int_{x=i}^{i+1} f(x)\ dx \le a_i.$$

- Summing up for i from 1 to $n-1$:

$$\sum_{i=1}^{n-1} a_{i+1} \le \int_{x=1}^{n} f(x)\ dx \le \sum_{i=1}^{n-1} a_i.$$

- Rearrange terms to get:

Fact 19
Suppose $f(i) = a_i$ for $i \in \{1, 2, \ldots\}$ and $f(x)$ is positive, continuous, and decreasing. Then

$$a_n + \int_{x=1}^{n} f(x)\ dx \le \sum_{i=1}^{n} a_i \le a_1 + \int_{x=1}^{n} f(x)\ dx$$

Example: Using an integral, bound

$$\sum_{j=1}^{n} j^3.$$

This isn't decreasing, but by using

$$\sum_{j=1}^{n} j^3 = \sum_{k=1}^{n} (n+1-k)^3,$$

now $f(x) = (n+1-x)^3$ is positive, continuous, and decreasing. So

$$\begin{aligned}\int_{x=1}^{n} (n+1-x)^3\ dx &= -(n+1-x)^4/4|_1^n \\ &= n^4/4 - 1/4\end{aligned}$$

Here $k_1 = n^3$ and $k_n = 1$, so

$$\frac{n^4}{4} + \frac{3}{4} \leq \sum_{i=1}^{n} i^3 \leq \frac{n^4}{4} + n^3.$$

For instance, when $n = 10$ this gives:

$$2500 \leq \sum_{j=1}^{10} j^3 \leq 3500.$$

The proof of the Integral test uses an advanced fact from real analysis.

Fact 20
Suppose $a_i \geq 0$ for all i. Then either $\sum_i a_i = \infty$ or $\sum_i a_i = L$ for some finite L.

Proof of Integral Test. Using the inequalities above, if $\int_{x=1}^{\infty} f(x)\ dx = \infty$, then $\sum_{i=1}^{\infty} a_i = \infty$. And if $\int_{x=1}^{\infty} f(x)\ dx$ is finite, then $\sum_{i=1}^{\infty} a_i$ is bounded above by a constant so can't be ∞. Hence by the previous fact $\sum a_i$ converges. $\square$

26 Series tests: alternating

Question of the Day Does $\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}$ converge?

Today

- Alternating series
- Bounding the series using partial sums

The Harmonic series

- The series $1 + 1/2 + 1/3 + \cdots$ is the *harmonic series*
- This series *diverges* by the p-series test
- $1 - 1/2 + 1/3 - 1/4 + \cdots = \sum (-1)^{i-1}/i$ is called the *alternating harmonic series* because its terms alternate in sign.
- Does this series converge?

Definition 61
$\sum a_i$ is an **alternating series** if $a_{i+1}/a_i < 0$ for all i.

Look at the partial sums

- A *partial sum* of a series $\sum_{i=1}^{\infty} a_i$ is $S_n = \sum_{i=1}^{n} a_i$
- Look at the partial sums of the alternating harmonic series:

n	S_n
1	1
2	$1 - 1/2 = 1/2 = 0.5555\ldots$
3	$5/6 = 0.8333\ldots$
4	$7/12 = 0.5833\ldots$
5	$47/60 = 0.7833\ldots$
6	$37/60 = 0.6166\ldots$

- Note: $S_1 > S_3 > S_5$ and $S_2 < S_4 < S_6$
- This pattern holds when terms are decreasing in absolute value

Lemma 36
Suppose that $|a_{i+1}| \leq |a_i|$ for an alternating series and $a_s > 0$. Then $S_s \geq S_{s+2} \geq S_{s+4} \geq \cdots$ and $S_{s+1} \leq S_{s+3} \leq S_{s+5} \leq \cdots$.

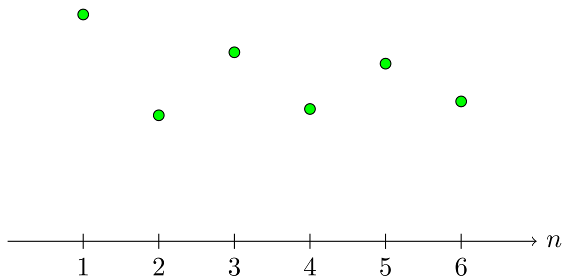
Proof. Let $k \in \{1, 2, 3, \ldots\}$. Then

$$S_{s+2k} - S_{s+2k+2} = \left[\sum_{i=s}^{s+2k} a_i\right] - \left[\sum_{i=s}^{s+2k+2} -2\right] = -a_{2+2k+1} + a_{2+2k+2}.$$

By assumption $|a_{2+2k+2}| \leq |a_{2+2k+1}|$, and since this is an alternating series $-a_{2+2k+1} > 0$, so $S_{s+2k} - S_{s+2k+2} \geq 0$. So $S_{s+2k} \geq S_{s+2k+2}$.

The proof that $S_{s+2k-1} \leq S_{s+2k+1}$ is similar. □

In picture form, for $a_1 > 0$, the odd S_n gives upper bounds on the series, while the even S_n give lower bounds



- Note $S_n - S_{n+1} = -a_{n+1}$.
- So upper and lower bounds converge to a single number if and only if $a_{n+1} \to 0$

Lemma 37 (Alternating series test)
An alternating series converges if and only if $a_{i+1} \to 0$ as $n \to \infty$.

- Remember, only works for alternating!
- Harmonic series $1 + 1/2 + \cdots$ an example of series whose terms go to 0 in magnitude, but which does not converge!

Qotd

- $\sum (-1)^{i-1}/i$ is an alternating series
- $\lim_{i \to \infty} (-1)^{i-1}/i = 0$
- Therefore it converges!

Absolute versus conditional convergence

- So for $a_i = (-1)^{i-1}/i$, $|a_i| = 1/i$
- Hence $\sum a_i$ converges, but $\sum |a_i|$ does not.

Lemma 38
If $\sum |a_i|$ converges, so does $\sum a_i$

Definition 62
If $\sum a_i$ converges and $\sum |a_i|$ converges, say that the series is **absolutely convergent**. If $\sum a_i$ converges but $\sum |a_i|$ diverges, say that the series is **conditionally convergent**.

- The alternating harmonic series is conditionally convergent
- Why does it matter?
- For a series that is absolutely convergent, you can rearrange the terms and you get the same answer
- For a series that is conditionally convergent, rearranging the terms can get you a different answer!

Original terms	1	$-1/2$	$1/3$	$-1/4$	$1/5$	$-1/6$
New ordering	1	$-1/2$	$-1/4$	$1/3$	$-1/6$	$-1/8$

- How do
$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$
and
$$B = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots$$
compare?

- Note that $1 - 1/2 = 1/2$ and $1/3 - 1/6 = 1/6$, etcetera, so
$$B = 1/2 - 1/4 + 1/6 - 1/8 + \cdots = (1/2)(1 - 1/2 + 1/3 - 1/4) = A/2$$

- Just by rearranging the terms, have dropped the value of the series in half!

26.1 Factorials

A useful notation in mathematics is that of the factorial.

Definition 63
The **factorial** of a nonnegative integer n (written $n!$) is defined to be

- 1 when $n = 0$
- The product of the integers from 1 up to n when $n > 0$

Example
$$3! = 1 \cdot 2 \cdot 3 = 6, \quad 10! = 3\,628\,800$$
Also, $n!$ is the number of ways to arrange n different objects in a line.

Problem

- Find
$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$
to 4 sig figs

- Terms are decreasing in magnitude. Each partial sum gives an upper or lower bound on the sum. So just calculate partial sums until desired accuracy achieved.

n	S_n
0	1
1	0
2	0.500000
3	0.333333
4	0.375000
5	0.366666
6	0.368055
7	0.367857
8	0.367881

- To 4 sig figs, the value is $\boxed{0.3678}$.

Example

- Find upper and lower bounds on

$$1 - 5 + \frac{5^2}{2!} - \frac{5^3}{3!} + \frac{5^4}{4!} - \frac{5^5}{5!} + \frac{5^6}{6!} - \cdots$$

- Note that the terms at the beginning at getting larger in magnitude $5 > 1$ nd $5^2/2! > 5$ until we reach $5^5/5! > 5^6/6!$.
- At that point the factorial in the denominator divides by a number at each term which is larger than 5, so the term is getting smaller in magnitude.
- Here $S_{23} = 0.006737867$ and $S_{24} = 0.006737963$, so to 4 sig figs, the answer is $\boxed{0.006737}$.

27 Series tests: ratio

Question of the Day Does

$$(1/2) + 2(1/2)^2 + 3(1/2)^3 + 4(1/2)^4 + \cdots$$

converge or diverge?

Today

- Ratio test for series convergence
- Power series

Could use integral test

- Positive, decreasing, $a_i = i(1/2)^i$ is a continuous function in i.
- So could use Integral test
- Need to do integration by parts, so somewhat difficult
- Faster way is to use ratio test

Geometric series

- Recall a geometric series has the form

$$1 + r + r^2 + r^3 + \cdots$$

  - Converges if and only if $|r| < 1$
  - When it converges, adds up to $1/(1-r)$.
- There's a way to define geometric series using ratios

  Lemma 39
  A series $\sum a_i$ is a geometric series if and only if $a_{i+1}/a_i = r$ for all i.

- Qotd almost a geometric series with $r = 1/2$, but not quite
- $a_i = i(1/2)^i$, so

$$\frac{a_{i+1}}{a_i} = \frac{(i+1)(1/2)^{i+1}}{i(1/2)^i} = \left(1 + \frac{1}{i}\right)(1/2)$$

- Close enough to geometric with $r = 1/2$ to make it converge!

Lemma 40 (The ratio test)
Suppose $\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| = L$. Then if $L > 1$ the series diverges. If $L < 1$ the series absolutely converges. If $L = 1$ the test fails.

Qotd

- Since

$$\lim_{i\to\infty} \left(1 + \frac{1}{i}\right)(1/2) = 1/2 \in (-1, 1),$$

  the series converges.

Example

- For what values of x does $\sum_{i=0}^{\infty}(2x)^i$ converge?
- It always does if $x = 0$
- Suppose $x \neq 0$ Look at the ratio between successive terms:
$$\frac{a_{i+1}}{a_i} = \frac{(2x)^{i+1}}{(2x)^i} = 2x$$
- If $|2x| < 1$, then the series converges, if $|2x| > 1$ it diverges, and if $|2x| = 1$, the test fails.
- So for $|x| < 1/2$, the series converges, that is, $x \in (-1/2, 1/2)$.
- At $x = 1/2$, $1 + 1 + 1 + \cdots$ diverges
- At $x = -1/2$, $1 - 1 + 1 - 1 + \cdots$ diverges
- So the series converges if and only if $\boxed{x \in (-1/2, 1/2)}$

The ratio test is especially helpful when dealing with terms that contain factorials.

Using the ratio test on a factorial example

- Show that the series
$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
converges
- Before tackling that, look at some ratios of factorials:
$$\frac{6!}{7!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = \frac{1}{7}.$$
- In general:
$$\frac{n!}{(n+1)!} = \frac{1}{n+1}$$
- Now for the ratio test: $a_i = x^i/i!$
$$\frac{a_{i+1}}{a_i} = \frac{x^{i+1}/(i+1)!}{x^i/i!} = \frac{x^{i+1}}{x^i} \cdot \frac{i!}{(i+1)!} = \frac{x}{i+1}.$$
Since $\lim_{i \to infty} x/(i+1) = 0$ *no matter what x is* this series always converges.

27.1 Power series

Definition 64
A series of the form
$$\sum_{i=0}^{\infty} a_i(x-c)^i$$
for constant c is called a **power series**.

- A power series is like a polynomial that goes on forever.
- Always converges when $x = 0$
- $\sum(2x)^i = \sum 2^i x^i$ converges when $x \in (-1/2, 1/2)$
- $\sum x^i/i!$ converges for all real numbers

Lemma 41
For any power series, there are three possibilities

1: The power series only converges when $x = 0$.

2: There exists finite R such that the power series absolutely converges when $|x-c| < R$, and diverges when $|x-c| > R$.

3: The power series converges for all values of x.

Definition 65
The **radius of convergence** of a power series is

1: 0 when the series only converges at $x = 0$,

2: R when the series converges for $|x-c| < R$ and diverges for $|x-c| > R$,

3: ∞ when the series converges everywhere.

When $|x-c| < R$, say that x is **inside** the radius of convergence.

Examples

series	radius of convergence
$\sum i2^i x^i$	$1/2$
$\sum x^i/i!$	∞
$\sum i!x^i$	0

27.2 Swapping limits inside the radius of convergence

- The radius of convergence is important because if x is inside it, you can swap limit operators

Lemma 42
Say that $f(x) = \sum_{i=0}^{\infty} a_i(x-c)^i$ for $|x-c| < R$. Then for continuous functions g and h,

$$\begin{aligned}\lim_{w\to a} g(w)f(h(w)) &= \lim_{w\to a} g(w)\sum_{i=0}^{\infty} a_i(h(w)-c)^i \\ &= \sum_{i=0}^{\infty} a_i \lim_{w\to a} g(w)(h(w)-c)^i.\end{aligned}$$

- In particular, inside the radius of convergence, you can swap differentiation and summation
- Example: Recall

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad (\forall x \in (-1,1))$$

$$
\begin{aligned}
\left[\frac{1}{1-x}\right]' &= \lim_{h\to 0}\frac{(1/(1-(x+h)))-(1/(1-x))}{h} \\
&= \lim_{h\to 0}\sum_{i=0}^{\infty}\frac{(x+h)^i-x^i}{h} \\
&= \sum_{i=0}^{\infty}\lim_{h\to 0}\frac{(x+h)^i-x^i}{h} \\
&= \sum_{i=0}^{\infty}[x^i]' \\
&= \sum_{i=0}^{\infty} i x^{i-1}.
\end{aligned}
$$

Useful fact: $\sum_{i=0}^{\infty} ix^{i-1} = (1-x)^{-2}$ for $x \in (-1,1)$

- Comes up center of mass calculations physics
- Comes up in expected value in probability

What happens on the edge of the radius of convergence?

- What happens at $x \in \{-1, 1\}$ for
$$\sum_{i=1}^{\infty}\frac{x^i}{i}$$
- What happens at $x \in \{-1, 1\}$ for
$$\sum_{i=1}^{\infty}\frac{x^i}{i^2}$$

28 Power series and Taylor series

Question of the Day What is $\sum_{i=1}^{\infty}(1/2)^i/i$?

Today

- Analytic functions
- Differentiation and antidifferentiation of a power series
- Power series for elementary functions
- Taylor series

28.1 Analytic functions

Definition 66
A function that is equal to a power series is called **analytic**.

Example:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

for $x \in (-1,1)$. Hence $1/(1-x)$ is analytic.

28.2 Differentiation/Antidifferentiation of power series

- Differentiate polynomials term by term

$$[1 + x + 2x^2]' = [1]' + [x]' + [2x^2]' = 0 + 1 + 4x = 1 + 4x$$

- Antidifferentiate polynomials term by term

$$\text{ad}_x(1 + x + 2x^2) = x + x^2/2 + (2/3)x^3.$$

Lemma 43
For a power series $f(x) = \sum a_i(x-c)^i$ with nonzero radius of convergence R, if $|x-c| < R$, then

$$\left[\sum a_i(x-c)^i\right]' = \sum [a_i(x-c)^i]' = \sum i \cdot a_i(x-c)^{i-1}$$

and

$$\text{ad}_x\left(\sum a_i(x-c)^i\right) = \sum \text{ad}_x(a_i(x-c)^i) = \sum \frac{a_i(x-c)^{i+1}}{i+1}$$

1: In other words, you can swap the summation and derivative/antiderivative operators when working inside the radius of convergence.

2: Technically: $\sum a_i(x-c)^{i+1}/(i+1) \in \text{ad}_x(\sum a_i(x-c)^i)$.

Example

- What is $\sum_{i=1}^{\infty}(1/2)^i/i$?
- Let $f(x) = \sum_{i=1}^{\infty} x^i/i$. Then the goal is to find $f(1/2)$.

- What is the radius of convergence. Ratio test!

$$\lim_{i\to\infty} \frac{x^{i+1}/(i+1)}{x^i/i} = \lim_{i\to\infty} x\frac{i}{i+1} = x,$$

so converges when $|x| < 1$. Radius of convergence is 1.

- Differentiating gives:

$$f'(x) = \sum_{i=1}^{\infty} [x^i/i]' = \sum_{i=1}^{\infty} x^{i-1} = \sum_{j=0}^{\infty} x^j,$$

where the last equality was just $j = i - 1$.

Note that in a summation you need only change the limits and the summand, there is no differential.

- That last is a geometric series, so

$$f'(x) = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}.$$

- Now antidifferentiate both sides:

$$f(x) = \ln(1-x)/(-1) = -\ln(1-x).$$

- So $f(1/2) = -\ln(1/2) = \ln(2) \approx 0.6931$

The exponential function

- Recall

$$f(x) = 1 + x + x^2/2 + \cdots = \sum_{i=0}^{\infty} x^i/i!$$

has infinite radius of convergence

- What is $f'(x)$?

$$\begin{aligned}\left[1 + x + x^2/2! + x^3/3! + \cdots\right]' &= [1]' + [x]' + [x^2/2!]' + [x^3/3!]' + \cdots \\ &= 1 + x/1! + x^2/2! + \cdots = f(x).\end{aligned}$$

- So $f'(x) = f(x)$ and $f(0) = 1$. But from our earlier separation of variables argument, that has a unique solution $f(x) = e^x$.

- Hence

$$\exp(x) = 1 + x + x^2/2! + x^3/3! + \cdots$$

This gives us a way to rigorously define the common elementary functions: use power series!

Definition 67

The exponential (exp), sine (sin), and cosine (cos) functions are defined as follows

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Using an argument similar to the qotd, can show

Lemma 44
The power series for natural logarithm is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for $|x| < 1$.

28.3 Taylor series

Definition 68
A function that is equal to a power series is called **analytic**.

- Inside radius of convergence, can use power series to get derivatives.
- Suppose $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Then $f(0) = a_0$.
- What is $f'(0)$?

$$\begin{aligned}[f(x)]'|_{x=0} &= [a_0 + a_1 x + a_2 x^2/2! + a_3 x^3/3! + \cdots]'|_{x=0} \\ &= [a_1 + a_2 x/1! + a_3 x^2/2! + \cdots]|_{x=0} = a_1.\end{aligned}$$

- What is $f''(0)$?

$$\begin{aligned}[f'(x)]'|_{x=0} &= [a_1 + a_2 x/1! + a_3 x^2/2! + \cdots]'|_{x=0} \\ &= [a_2 + a_3 x/1! + a_4 x^2/2! + \cdots]|_{x=0} \\ &= a_2\end{aligned}$$

Let $f^{(i)}$ denote the ith derivative of f. ($f^{(0)} = f$.)

Lemma 45
If $f(x) = \sum_{i=0}^{\infty} a_i x^i$, then

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots,$$

or in other words, $a_i = f^{(i)}(0)/i!$.

Definition 69
Suppose $f \in C^\infty$. Then the **Taylor series** of f centered at c is

$$f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + \cdots = \sum_{i=0}^{\infty} f^{(i)}(c)\frac{(x-c)^i}{i!}.$$

When $c = 0$, this is also known as a **Maclaurin** series.

Lemma 46
If f is analytic, then f equals its Taylor series for all x in the radius of convergence.

Note

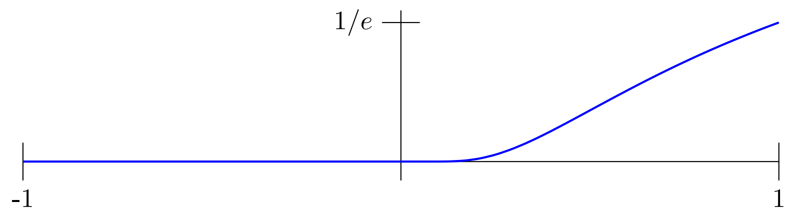
- The Taylor series of a function might only equal the function at $x = c$!

Example of a function that does not equal its Taylor series

- Consider the function

$$f(x) = e^{-1/x}\mathbb{1}(x > 0)$$

  A graph of the function from -1 to 1 looks like:



- This function is very flat at 0, in fact, $f^{(n)}(0) = 0$ for all n
- So the Taylor series is

$$0 + 0x + 0x^2/2! + 0x^3/3! + \cdots = 0.$$

  This has infinite radius of convergence
- But the function is not zero if $x > 0$, so the function does not equal the Taylor series inside the radius of convergence.

28.4 Finding power series

Example

- What is the power series for $\exp(-x^2/2)$?
- Plug $-x^2/2$ into the series for exp

$$1 - \frac{x^2}{2} + \frac{1}{2!} \cdot \frac{x^4}{2^2} - \frac{1}{3!} \cdot \frac{x^6}{2^3} = \sum_{i=0}^{\infty} \frac{x^{2i}}{2^i i!}$$

- Can build on previous work
- What is the power series for $x \exp(-x^2/2)$?

$$\begin{aligned} x\left(1 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots\right) &= x - x^3/2 + x^4/(2^2 \cdot 2!) \\ &= \sum_{i=0}^{\infty} \frac{x^{2i+1}}{2^i i!}. \end{aligned}$$

29 Taylor Polynomials

Question of the Day What is the best quadratic approximation to $\exp(x)$ at $x = 1$?

Today

- Approximating functions with polynomials
- Bounding the error

Recall

- The derivative is the slope of the line that best approximates $f(x)$ near $(c, f(c))$.

$$f(x) = f(c) + f'(c)(x - c) + o(x - c).$$

- Let $T_1(x) = f(c) + f'(c)(x - c)$.
- Then $T_1(c) = f(c)$, and $T_1'(x) = f'(c) = T_1'(c)$, so T_1 matches both f and f' at $x = c$.
- The best fit quadratic approximation can do better: can match f, f', and f'' at $x = c$.
- Write the quadratic as
$$T_2(x) = f(c) + f'(c)(x - c) + a_2(x - c)^2.$$
- What should a_2 be in order that $T_2''(c) = f''(c)$?

Answer

- First we need to calculate the second derivative

$$\begin{aligned} T_2''(x) &= [[T_1(x)]']' \\ &= [f'(c) + 2a_2(x - c)]' \\ &= 2a_2 \end{aligned}$$

  To make $2a_2 = f''(c)$, $a_2 = f''(c)/2$.
- For $f(x) = e^x$, $f'(x) = f''(x) = e^x$, so $f'(1) = f''(1) = e^1 = e$.
- So best quadratic approximation to $\exp(x)$ at $x = 1$ is:

$$\boxed{T_2(x) = e + e(x - 1) + (e/2)(x - 1)^2}$$

- Try plotting this in WA

```
plot exp(x) and (e + e(x-1)+e(x-1)^2) from -1 to 2
```

- Could also write in $ax^2 + bx + c$ form:
$$\boxed{(e/2)x^2 + e/2}$$

What if we have a cubic?

- What $T_3(x)$ such that $T_3(c) = f(c)$, $T_3'(c) = f'(c)$, $T_3''(c) = f''(c)$ and $T^{(3)}(c) = f^{(3)}(c)$.
- Use $T_2(x)$ to get first three identities:

$$T_3(x) = T_2(x) + a_3(x-c)^3$$

- Now take third derivative:

$$\begin{aligned} [[[T_3(x)]']']' &= [[[T_2(x) + a_3(x-c)^3]']']' \\ &= [[T_2'(x) + 3a_3(x-c)^2]']' \qquad\qquad = [T_2''(x) + 3\cdot 2a_3(x-c)]' \\ &= T_2^{(3)}(x) + 3\cdot 2\cdots 1a_3 \end{aligned}$$

- Note T_2 is a quadratic, so the third derivative is 0 [In general, for a polynomial of degree n, the mth derivative where $m > n$ is zero.]
- So $T_3^{(3)}(x) = 3!a_3$, hence $a_3 = f^{(3)}(c)/3!$
- Can generalize this arugment

Definition 70
For a function $f \in C^n((a,b))$ and $c \in (a,b)$, let

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f'(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}$$

be the **nth degree Taylor polynomial**

Lemma 47 (Taylor polynomials)
For a function $f \in C^n$, the nth degree Taylor polynomial T_n is the unique polynomial of degree n such that $(\forall i \in \{0,\ldots,n\})(T_n^{(i)}(c) = f^{(i)}(c))$

Factorials

- The nth derivative of $(x-c)^n$ is $n\cdot(n-1)\cdot(n-2)\cdots 1$
- That's where the $n!$ in the Taylor polynomials is coming from

Relation to Taylor series

- $\lim_{n\to\infty} T_n(x) = T(x)$, the Taylor series expansion.

29.1 Applications of power/Taylor series

- Taylor series can be used to approximately solve DE's
- Consider the following second order DE:

$$y'' - x^2 y = e^x.$$

- Try to find the power series for the left hand side, and the right hand side. Let

$$y = \sum_{i=0}^{\infty} a_i x^i.$$

- Inside the radius of convergence

$$\begin{aligned}
-x^2 y &= \sum_{i=0}^{\infty} a_i x^{i+2} \\
&= a_0 x^2 + a_1 x^3 + \cdots \\
y'(x) &= \sum_{i=1}^{\infty} i a_i x^{i-1} \\
y''(x) &= \sum_{i=2}^{\infty} i(i-1) a_i x^{i-2} \\
&= 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots \\
e^x &= 1 + x + x^2/2 + x^3/6 + \cdots
\end{aligned}$$

- If I say $(a+b)x^2 + (b+c)x + b = 3x^2 + 2x + 1$, then you know

$$a + b = 3, \quad b + c = 2, \quad b = 1.$$

- This is called *coefficient matching*
- Same works for power series
- For $y'' - x^2 y = e^x$, it must be true that

$$\begin{aligned}
2a_2 &= 1 \\
6a_3 &= 1 \\
12a_4 + a_0 &= 1/2 \\
20a_5 + a_1 &= 1/6 \\
30a_6 + a_2 &= 1/24 \\
42a_7 + a_3 &= 1/120
\end{aligned}$$

  ...and so on.

- This is an underdertermined set of equations

$$y(x) = a_0 + a_1 x + (1/2)x^2 + (1/6)x^3 + (((1/2) - a_0)/24)x^4 + \cdots$$

- Suppose you are told $y(0) = y'(0) = 0$. Then the power series solution is:

$$y(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{48}x^4 + \cdots$$

  This is very accurate near 0.

Quickly finding limits

- What is

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}?$$

- Remember $\cos(x) = 1 - x^2/2 + x^4/4! - \cdots$
- So $1 - \cos(x) = x^2/2 - x^4/4! + \cdots$
- The radius of convergence is ∞, so

$$\frac{1 - \cos(x)}{x^2} = \frac{1}{2} - \frac{1}{4!}x^2 + \cdots .$$

- Taking the limit as x goes to 0 gives $\boxed{1/2}$

30 Finding the error in Taylor polynomial approximations

Question of the Day Bound $\exp(1)$ above and below using the fourth degree Taylor polynomial

Today

- Error with alternating Taylor series
- Error in Taylor polynomials

What is error?

- If $\hat{y}$ approximates y, then the error in the approximation is $y - \hat{y}$.
- Typically care about *absolute error*, $|y - \hat{y}|$
- Sometimes care about *relative error*, $|(y - \hat{y})/y|$

30.1 Bounding with Taylor series for alternating series

- Recall that for an alternating series whose terms are decreasing in magnitude, can bound using odd or even sums.
- Bound $\ln(3/2)$ using the second and third degree Taylor polynomials for $\ln(1 + x)$
- Recall for $x \in (-1, 1)$,

$$\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6 + \cdots .$$

For $x \in [0, 1)$, terms alternate and are decreasing. So

$$x - x^2/2 \leq \ln(1 + x) \leq x - x^2/2 + x^3/3$$

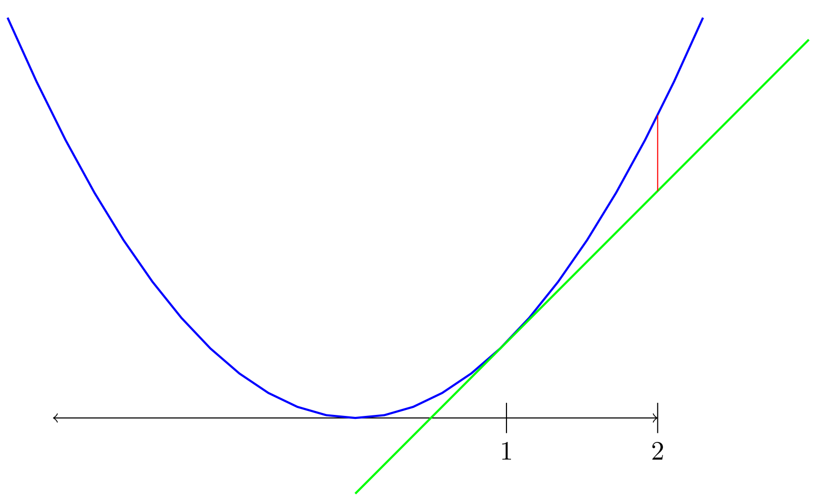
which gives $\ln(1 + 1/2) \in [3/8, 5/12]$, $\boxed{\ln(3/2) \in [0.3750, 0.4167]}$

- Note, when finding answer to 4 sig figs, have to round final digit down for lower bound, and up for upper bound.

30.2 Finding the error in a general Taylor series

Error in a linear approximation

- Look at how well a line approximates a parabola:



- The greater the second derivative of the parabola is, the farther it pulls away from the line.
- If the second derivative was 0, it would be a line!
- This motivates the following result

Lemma 48
Let $f \in C^2((a,b))$ and $c \in (a,b)$. Then $T_1(x) = f(c) + f'(x)(x-c)$. Then there exists a function $R_n(x)$ such that

$$f(x) = T_1(x) + R_1(x),$$

where there is some α between x and c such that

$$R_1(x) = \frac{f''(\alpha)}{2!}(x-c)^2.$$

- Apply this result to $\exp(x)$ when $x = 1$

i	$f^{(i)}(x)$
0	$\exp(x)$
1	$\exp(x)$
2	$\exp(x)$

- For $c = 0$, the result says that

$$\exp(x) = \exp(0) + \exp(0)x + R_1(x) = 1 + x + R_1(x),$$

  where $R_1(x) = \exp(\alpha)x^2/2!$ for some α between 0 and x.
- Suppose $x > 0$. Then $\alpha \in [0,x]$, and since $\exp(\alpha)$ is increasing in α,

$$\min_{\alpha \in [0,\alpha]} R_1(x) = x^2/2, \quad \max_{\alpha \in [0,\alpha]} R_1(x) = e^x x^2/2,$$

- Hence for $x > 0$,

$$1 + x + \frac{x^2}{2} \leq \exp(x) \leq 1 + x + \frac{e^x x^2}{2}.$$

- So when $x = 1$,

$$2.5 \leq e^1 \leq 2 + e^1(1/2).$$

- Note upper bound involves e^1, but that is what we are trying to find. Treat e^1 as an unknown to eliminate it from both sides:

$$e^1(1 - 1/2) \leq 2 \Rightarrow e^1 \leq 2/(1 - 1/2) = 4.$$

  So

$$\boxed{2.500 \leq e^1 \leq 4}$$

General result

- Last item presented it for T_1 and R_1, generalizes

Lemma 49
Let $f \in C^{n+1}((a,b))$ and $c \in (a,b)$. Then for

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i,$$

let

$$R_n(x) = f(x) - T_n(x).$$

Then for each x there exists an α between x and c such that

$$R_n(x) = \frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-c)^{n+1}.$$

Qotd

- The fourth degree Taylor polynomial for $\exp(x)$ is

  $$T_4(x) = 1 + x + x^2/2! + x^3/3! + x^4/4!.$$

  The difference between this and $\exp(x)$ is $R_4(x)$, where

  $$R_4(x) = \frac{\exp(\alpha)}{5!}x^5.$$

  where α is between 0 and x.

- Suppose $x > 0$, then $\exp(\alpha) \in [1, e^x]$. So

  $$T_4(1) + 1/5! \leq e^1 \leq T_4(1) + e^1 1/5!.$$

  Since $T_4(1) = 65/24$,

  $$e^1 \leq 65/24 + e^1(1/120) \to e^1 \leq (65/24)/(119/120)$$

  so

  $$e^1 \in [163/60, 325/119] \in [2.716, 2.732].$$

30.3 Writing the error using Big-O notation

Recall

- If $\lim_{x \to c} f(x)/g(x) < \infty$, say $f(x)$ is Big-O of $g(x)$.

Lemma 50
Let $f \in C^{n+1}((a,b))$ and $c \in (a,b)$. Then

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i + O((x-c)^{n+1}).$$

- Less helpful than actually calculating error bounds explicitly
- Still, gives idea of how error grows as x moves away from c.

Example

- What is the fourth degree Taylor polynomial (with error in Big-O notation) for $\sin(x)$?
- Answer: The Taylor series is

  $$x - x^3/3! + x^5/5! - x^7/7! + \cdots$$

  so the fourth degree Taylor polynomial is

  $$x - x^3/3!$$

  (note that the x^4 term has a 0 coefficient in front of it.)
- The error term is then $O(x^5)$. Hence

  $$\boxed{x - x^3/3! + O(x^5)}$$

  is the answer.

31 Numerical methods for integration

Question of the Day Approximately what is $\int_{x=0}^{1} \exp(-x^2/2)\ dx$?

Today

- Constant Rule
- Trapezoidal Rule
- Simpson's Rule

Solving integrals analytically versus numerically

- Sometimes we can find an antiderivative of a function explicity
- When that happens, can solve an integral *analytically* to find the exact solution
- Analytic solutions can include variables in the integral. Example:

$$\int_0^1 \lambda \exp(-\lambda x)\ dx = 1 - \exp(-\lambda x)$$

- Otherwise, have to solve the integral *numerically*.
  - Get a number at the end
  - Only an approximation, never the exact answer
  - Doesn't work with variables

The idea

- Take a complicated function (like $\exp(-x^2/2)$)
- Replace it with a function that we know how to integrate
  - Like a line
  - Or a parabola
- Integrate the approximation of the function instead

31.1 Left Endpoint Rule

- Simplest approximation $\exp(-x^2/2) = \exp(-0^2/2)$ for all $x \in [0, 1]$.

$$\int_{x=0}^{1} \exp(-x^2/2)\ dx \approx \int_{x=0}^{1} \exp(-0^2/2)\ dx = 1(1-0) = 1.$$

- More generally:

$$\int_{x=a}^{b} f(x)\ dx \approx \int_{x=a}^{b} f(x)\ dx = f(a)(b-a).$$

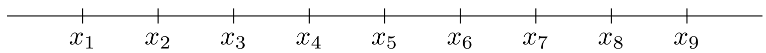
- Better if you break into more than one interval

$$\begin{aligned}\int_{x=0}^{1} \exp(-x^2/2)\ dx &= \int_{x=0}^{1/2} \exp(-x^2/2)\ dx + \int_{x=1/2}^{1} \exp(-x^2/2)\ dx \\ &\approx \exp(-0^2/2)(1/2-0) + \exp(-(1/2)^2/2)(1-1/2) \\ &\approx \frac{1-0}{2}\left[\exp(-0^2/2) + \exp(-(1/2)^2/2)\right] \\ &= 0.9412\ldots\end{aligned}$$

Definition 71 (Left Endpoint Rule)
For $a < b$, and $i \in \{0, 1, \ldots, n\}$, let $x_i = a + i(b-a)/n$. Then the **left endpoint rule** approximation for $\int_a^b f(x)\, dx$ is

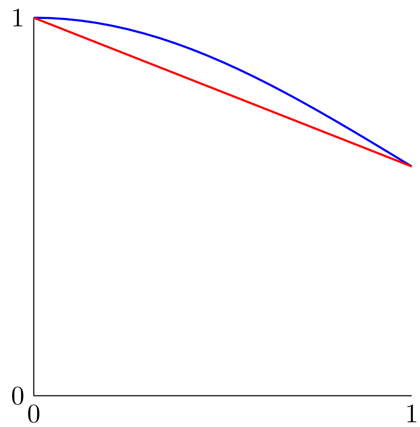
$$\hat{I}_{\text{trap}} = \frac{b-a}{n}\left[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right].$$

In general, break the line up into n intervals of equal length



Qotd

- Use a line to approximate $\exp(-x^2/2)$;



$$\exp(-x^2/2) \approx 1 + (e^{-1/2} - 1)x$$

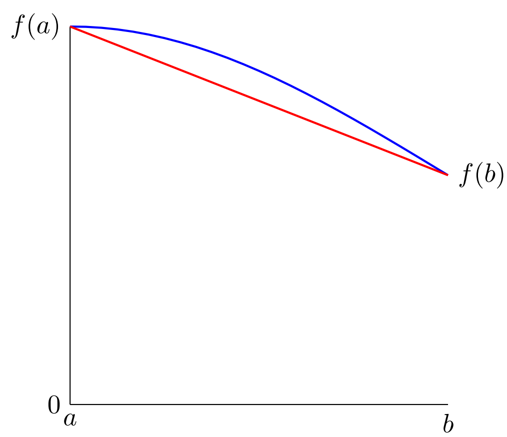
So

$$\begin{aligned}\int_{x=0}^{1} \exp(-x^2/2)\, dx &\approx \int_{x=0}^{1} 1 + (e^{-1/2}-1)x\, dx \\ &= x + (e^{-1/2}-1)x^2/2|_0^1 \approx 0.8032\end{aligned}$$

- Natural question to ask: How good is this approximation? Next time...

31.2 Trapezoidal Rule

General formula



- Want line $c_1 + c_2 x$ to pass through $(a, f(a))$ and $(b, f(b))$:

$$\begin{aligned} c_1 + c_2 a &= f(a) \\ c_2 + c_2 b &= f(b) \end{aligned}$$

  Solve to get

$$c_1 = (bf(a) - af(b))/(b-a), \quad c_2 = (f(b) - f(a))/(b-a).$$

- So

$$\begin{aligned} \int_a^b f(x)\ dx &\approx \int_a^b \frac{f(b)-f(a)}{b-a}x + \frac{bf(a)-af(b)}{b-a}\ dx \\ &= [f(b)-f(a)]\frac{b^2-a^2}{2(b-a)} + bf(a) - af(b) \\ &= \frac{1}{2}[f(b)-f(a)](b+a) + bf(a) - af(b) \\ &= \frac{f(b)+f(a)}{2}(b-a) \end{aligned}$$

- This is the area of a trapezoid when $f(a)$ and $f(b)$ are at least 0, so call this the trapezoidal rule:

$$\int_a^b f(x)\ dx \approx \frac{f(a)+f(b)}{2}(b-a)$$

Improving the rule

- The smaller the interval, the better the line approximates the function.
- Break the integral into smaller intervals to improve the approximation.
- Example:

$$\begin{aligned} \int_{x=0}^1 \exp(-x^2/2)\ dx &= \int_{x=0}^{1/2} \exp(-x^2/2)\ dx + \int_{x=1/2}^1 \exp(-x^2/2)\ dx \\ &\approx \frac{\exp(-(1/2)^2/2) + \exp(0)}{2}(1/2 - 0) + \frac{\exp(-1^2/2) + \exp(-(1/2)^2/2)}{2}(1 - 1/2) \\ &\approx 0.8428 \end{aligned}$$

- Using more intervals:

n	$\hat{I}$
1	0.8032
2	0.8428
3	0.8499
$\vdots$	$\vdots$
∞	0.85556

- To fit n intervals between a and b, set $x_0 = a$, $x_n = b$, width of interval is $w = (b-a)/n$, $x_1 = x_0 + w$, $x_2 = x_0 + 2w$, and so on.

Definition 72 (Trapezoidal Rule)
For $a < b$, and $i \in \{0, 1, \ldots, n\}$, let $x_i = a + i(b-a)/n$. Then the **Trapezoidal rule** approximation for $\int_a^b f(x)\ dx$ is

$$\hat{I}_{\text{trap}} = \frac{b-a}{n}\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right].$$

Example

- Estimate $\int_{\pi}^{2\pi} \sin(x)/x\ dx$ using the Trapezoidal Rule with 4 intervals.
- Width of an interval $(2\pi - \pi)/4 = \pi/4$.
- Interval endpoints $(4/4)\pi < (5/4)\pi < (6/4)\pi < (7/4)\pi < (8/4)\pi$

x_i	$f(x_i)$
π	0
$(5/4)\pi$	-0.18006324
$(6/4)\pi$	-0.21220659
$(7/4)\pi$	-0.12861662
2π	0

- So approximation

$$\frac{2\pi - \pi}{4}\left[(1/2)0 - 0.18006 - 0.2122 - 0.1286 + (1/2)0\right] \approx \boxed{\text{-0.4091}}$$

- Exact answer -0.4337

31.3 Simpson's Rule

- Instead of using lines, can use parabolas when n is even.
- Gives a rule that is usually more accurate.

Definition 73 (Simpson's Rule)
For $a < b$, and $i \in \{0, 1, \ldots, n\}$, let $x_i = a + i(b-a)/n$. Then the **Simpson's rule** approximation for $\int_a^b f(x)\ dx$ is

$$\hat{I}_{\text{Simp}} = \frac{b-a}{n}\left[\frac{1}{3}f(x_0) + \frac{4}{3}f(x_1) + \frac{2}{3}f(x_2) + \cdots + \frac{4}{3}f(x_{n-1}) + \frac{1}{3}f(x_n)\right].$$

- Using Simpson's rule:

$$\int_{\pi}^{2\pi} \frac{\sin(x)}{x}\ dx \approx \frac{\pi}{4}\left[(4/3)(-0.1800) + (2/3)(-0.2122) + (4/3)(-0.1286)\right] \approx -0.4343$$

- Better estimate with same amount of computation!

32 Error in numerical methods for integration

Question of the Day Bound $\int_0^1 \exp(-x^2/2)\, dx$ using the Trapezoidal rule

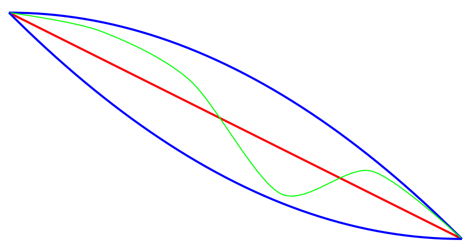
Today

- Error bounds on trapezoidal rule approximation for integrals

32.1 Error of Trapezoidal rule

Idea

- Get an upper and lower bound on f from a to b
- How big can f be an still pass through $f(a)$ and $f(b)$?
- Suppose $f''(x) \geq m$. Then f must lie below the parabola $p(x)$ where $p''(x) = m$, $p(a) = f(a)$, $p(b) = f(b)$.
- Suppose $f''(x) \leq M$. Then f must lie above the parabola $q(x)$ where $q''(x) = M$, $q(a) = f(a)$, $q(b) = f(b)$.



- Then its just algebra to calculate the integral of the upper parabola and the lower parabola

Lemma 51
Suppose $f \in C^2[a,b]$ and $f''(x) \in [m, M]$ for all $x \in [a,b]$. Then

$$\hat{I}_{\text{trap}} - \frac{(b-a)^3}{12} M \leq \int_a^b f(x)\, dx \leq \hat{I}_{\text{trap}} - \frac{(b-a)^3}{12} m$$

Lemma to algorithm

- This lemma gives an algorithm for bounding integrals
  1: Find the second derivative of f
  2: Maximize and minimize f over $[a,b]$ (Note that involves finding the third derivative of f to get critical points!)
  3: Use the bounds of the lemma to bound the integral

Note

- Result generalizes to handle case where divide $[a,b]$ into n intervals.
- $b - a \to (b-a)/n$, so $(b-a)^3 \to (b-a)^3/n^3$.
- Adding the inequalities for n different subintervals multiplies by n

Lemma 52 (Trapezoidal rule error)
Suppose $f \in C^2[a,b]$ and $f''(x) \in [m, M]$ for all $x \in [a,b]$. Then

$$\hat{I}_{\text{trap}} - \frac{(b-a)^3}{12n^2}M \leq \int_a^b f(x)\ dx \leq \hat{I}_{\text{trap}} - \frac{(b-a)^3}{12n^2}m$$

- The number of points to evaluate for trapezoid rule is $n+1$
- Since the error decreases inversely with n^2, call this a quadratic method

Qotd

- Find second (and third) derivative

$$\begin{aligned}
f(x) &= \exp(-x^2/2) \\
f'(x) &= \exp(-x^2/2)[-x^2/2]' = -x\exp(-x^2/2) \\
f''(x) &= [-x]'\exp(-x^2/2) + (-x)[\exp(-x^2/2)]' \\
&= -\exp(-x^2/2) + x^2\exp(-x^2/2) \\
&= (x^2-1)\exp(-x^2/2) \\
f'''(x) &= [x^2-1]'\exp(-x^2/2) + (x^2-1)[\exp(-x^2/2)]' \\
&= 2x\exp(-x^2/2) + (x^2-1)(-x)\exp(-x^2/2) \\
&= \exp(-x^2/2)[3x - x^3]
\end{aligned}$$

- Since $\exp(-x^2/2) > 0$, zeros when $3x - x^3 = 0$, $x = 0$, $x \in \{-\sqrt{3}, \sqrt{3}\}$.
- No critical points inside $[0,1]$
- $f''(0) = -1$, $f''(1) = 0$
- $\hat{I}_{\text{trap}} = (1 + \exp(-1/2))/2 = 0.803265329$
- Result:

$$\boxed{0.8032 \leq \int_0^1 \exp(-x^2/2)\ dx \leq 0.8866}$$

32.2 Error of Simpson's rule

- Can do the same sort of thing with Simpson's rule
- Bound the difference between f and a parabola that passes through $(a, f(a))$ and $(b, f(b))$ using the fourth derivative

Lemma 53 (Simpson's rule error)
Suppose $f \in C^4[a,b]$ and $f^{(4)}(x) \in [m, M]$ for all $x \in [a,b]$. Then

$$\hat{I}_{\text{Simp}} - \frac{(b-a)^5}{180n^4}M \leq \int_a^b f(x)\ dx \leq \hat{I}_{\text{Simp}} - \frac{(b-a)^5}{180n^4}m$$

- Error goes down as $1/n^4$, fourth order method
- Is it better than Trapezoidal rule?
- Yes if $f^{(4)}$ small in magnitude compared to $f^{(2)}$...

- ...and n is large compared to $b-a$
- To apply it to the question of the day, need fourth and fifth derivatives of $\exp(-x^2/2)$:

$$[\exp(-x^2/2)]^{(4)} = \exp(-x^2/2)(x^4 - 6x^2 + 3)$$
$$[\exp(-x^2/2)]^{(5)} = \exp(-x^2/2)(-x^5 + 10x^3 - 15x)$$

- Setting $-x(x^4 + 10x^2 - 15) = 0$ gives $x = 0$ as a solution, making $w = x^2$ gives $w^2 + 10w - 15 = 0 \Rightarrow w = -5 \pm 2\sqrt{10}$
- Since $w = x^2$ only the positive root matters, $x^2 = -5 + 2\sqrt{10}$, so $x = \sqrt{-5 + 2\sqrt{10}} \approx 1.150$ so out of the interval
- $f^{(4)}(0) = 3$, $f^{(4)}(1) = -2$, $\hat{I}_{\text{Simp}} = 0.856086$

$$\boxed{0.8394 \leq 0.8672}$$

32.3 Fixing the error bound

- Suppose I want an estimate $L \leq I \leq U$ such that $|U - L| \leq \epsilon$ for some tolerance ϵ How large does n need to be?
- Example: for qotd get bounds within 0.001 of each other
- For Trapezoidal rule

$$\text{upper bound} - \text{lower bound} = \frac{(b-a)^3}{12n^2}[M - m]$$

  Want

$$\frac{1}{12n^2} \leq 0.001 \Rightarrow n > \sqrt{(1/12)/0.001} = 9.12\ldots.$$

  So for Trapezoidal rule, $\boxed{n = 10}$ suffices (note n must be rounded *up* to guarantee error bound

- For Simpson's rule

$$\text{upper bound} - \text{lower bound} = \frac{(b-a)^5}{180n^4}[M - m]$$

  Want

$$\frac{1}{180n^4}5 \leq 0.001 \Rightarrow n > ((5/180)/0.001)^{1/4} = 2.295\ldots.$$

  So for Simpson's rule, $n = 4$, (can't use $n = 3$ with Simpson's, the number of intervals *must* be even)

33 First order linear DE's

Question of the Day Consider the following model for spread of disease:

$$ds = ks(m - s)\ dt$$

where k is a constant rate of growth, and m is the maximum population. Find $s(t)$.

Today

- Using partial fractions to solve DE's

Recall: separable DE's

$$\begin{aligned} ds = ks\ dt \Leftrightarrow \frac{1}{s}\ ds &= k\ dt \\ \ln(s) + C_s &= kt + C_t \\ \ln(s) &= kt + C \\ s &= e^{kt}e^{C} \\ &\boxed{s = s_0 e^{kt}}. \end{aligned}$$

- This was exponential growth.

Logistic model

- Growth fast when s far away from m
- Growth slows to 0 as s approaches m
- New DE called *logistic model*
- Try the same thing for new model.

$$\begin{aligned} ds &= ks(m - s)\ dt \\ \frac{1}{s(m-s)}\ ds &= k\ dt. \end{aligned}$$

- Can use partial fractions to antidifferentiate LHS

$$\begin{aligned} \frac{1}{s} \cdot \frac{1}{m-s} &= \frac{c_1}{s} + \frac{c_2}{m-s} \\ 1 &= c_1(m-s) + c_2 s. \end{aligned}$$

- Put in $s = m$ to get $c_2 = 1/m$ and $s = 0$ to get $c_1 = 1/m$.
- So

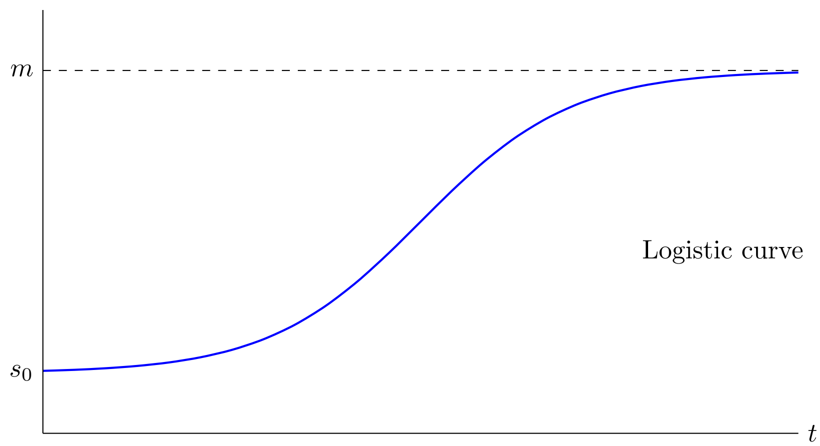
$$\frac{1}{s(m-s)}\ ds = \frac{1}{m}\left[\frac{1}{s} + \frac{1}{m-s}\right]\ ds = k\ dt.$$

- Hence

$$\begin{aligned} kt + C_t &= \frac{1}{m}\left[\ln(s) - \ln(m-s)\right] + C_s \\ &= \frac{1}{m}\ln\left(\frac{s}{m-s}\right) + C_s \\ mkt + C &= \ln\left(\frac{s}{m-s}\right) \\ \exp(mkt+C) &= s/(m-s) \\ (m-s)\exp(mkt+C) &= s \\ m\exp(mkt+C) &= s(1+\exp(mkt+C) \\ s &= \frac{m\exp(mkt)\exp(C))}{1+\exp(mkt)\exp(C)}. \end{aligned}$$

Let $\exp(C) = A$, then our solution is:

$$\boxed{s = \frac{mAe^{mkt}}{1+Ae^{mkt}}.}$$



- Unfortunately, not all de's are separable. For example:

$$dy = \left(e^{-x} - 2y\right)\, dx.$$

- Can also use FTC to write this:

$$\frac{dy}{dx} = y' = e^{-x} - 2y.$$

Definition 74
A **first order linear differential equation** is of the form

$$y' + P(x)y = Q(x).$$

- First order means there is y' but no y'' or y''' etcetera.
- Linear means that $P(x)$ multiplies y but not y^2 or other nonlinear function.
- The following theorem gives the solution to all first order linear differential equations.

Theorem 7 (First order linear DE's)
Consider $y' + P(x)y = Q(x)$. All solutions to this DE are of the form

$$y = \frac{1}{I(x)} \left[\mathrm{ad}_x(I(x)Q(x)) + C \right],$$

where $I(x) = e^{\mathrm{ad}_x(P(x))}$.

Example:

$$\text{Solve } y' = e^{-x} - 2y.$$

- First get in same form as formula:
$$y' + 2y = e^{-x}.$$

- Identify $P(x)$ and $Q(x)$.
$$P(x) = 2, \quad Q(x) = e^{-x}.$$

- Calculate $I(x)$.
$$I(x) = \exp(\mathrm{ad}_x(P(x))) = \exp(2x).$$

- Use formula:
$$\begin{aligned} y(x) &= \frac{1}{\exp(2x)} \left[\mathrm{ad}_x(\exp(2x)\exp(-x)) + C) \right] \\ &= \exp(-2x)[\exp(x) + C] \\ &= \boxed{e^{-x} + Ce^{-2x}}. \end{aligned}$$

Proof. Let $y = \left[\mathrm{ad}_x(I(x)Q(x)) + C \right] / I(x)$, and put it in DE.

$$\begin{aligned} y' &= I(x)^{-2}(-1)I'(x) \left[\mathrm{ad}_x(I(x)Q(x)) + C \right] + I(x)^{-1} \left[I(x)Q(x) \right] \\ &= -I(x)^{-1}I'(x) \left[\mathrm{ad}_x(I(x)Q(x) + C \right] / I(x) + Q(x) \\ &= -I(x)^{-1}I'(x)y + Q(x). \end{aligned}$$

Now

$$I'(x) = \left[\exp(\mathrm{ad}_x(P(x))) \right]' = P(x)\exp(\mathrm{ad}_x(P(x))) = P(x)I(x).$$

So

$$y' = -I(x)^{-1}P(x)I(x)y + Q(x),$$

which means $y' + P(x)y = Q(x)$.

Showing that this is the only solution is much more difficult! □

Example:

$$\text{Solve } x\ dy = \left(xe^{2x} - y \right)\ dx, \ x > 0.$$

- First get in proper form:
$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x}(xe^{2x} - y) \\ y' &= e^{2x} - y/x \\ y' + (1/x)y &= e^{2x}. \end{aligned}$$

- So $P(x) = 1/x$, $Q(x) = e^{2x}$.

- Hence $\mathrm{ad}_x(P(x)) = \ln(x)$, $I(x) = e^{\ln(x)} = x$.

- Using formula:

$$\begin{aligned} y &= \frac{1}{x}[\mathrm{ad}_x(xe^{2x}) + C] \\ &= \boxed{\frac{1}{x}\left[(2x-1)e^{2x} + C\right]}. \end{aligned}$$

34 Euler's Method for numerical differentiation

Question of the Day For

$$y'(x) - x = \ln(y),$$

if the solution passes through the point $(4, 6)$, approximately what is the y value at $x = 4.3$?

Today

- Numerical solution of differential equations

Classifying the problem

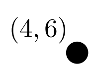
- This is a differential equation
- It is first order, since it contains only a first derivative
- It is nonlinear, since it is can't be put in the form $y' + P(x)y = Q(x)$ for functions P and Q
- Can't use our first order linear de solution
- Can however, solve it numerically.
- Simplest numerical method called *Euler's method*
- Euler did not invent, but did analyze the method

34.1 Using Euler's method

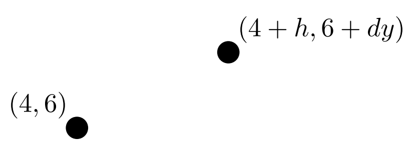
Consider the differential equation:

$$dy = f(x, y(x))\ dy.$$

For instance, if $y' - x = e^y$, then $y' = x + e^y$, so $f(x, y(x)) = x + e^y$. Now suppose that $x = 4$ and $y = 6$, so we are at the point $(4, 6)$ in the $x - y$ plane.



Now suppose we move a tiny bit h in the x direction. How much should y change?



Recall that

$$dy = f(x, y(x))\ dx,$$

so a good approximation for dy is

$$\hat{dy} = f(x, \hat{y}(x)) \cdot h.$$

This is the basis of Euler's Method:

Definition 75
Suppose $y' = f(x, y(x))$. Then **Euler's approximation** follows the rule:

$$\hat{y}(x + h) = \hat{y}(x) + h \cdot f(x, \hat{y}(x)).$$

Qotd

- First put it in the right form:
$$y'(x) = x + \ln(y)$$
- Next, decide on h. Suppose $h = 0.3$.
- It is important to know at least one point. In this case we know that $y(4) = 6$, so set $\hat{y}(4) = 6$.
- Apply the formula with $x = 4$:
$$\begin{aligned}\hat{y}(x+h) &= \hat{y}(x) + h \cdot f(x, \hat{y}(x)) \\ \hat{y}(4+0.3) &= 6 + (0.3)[4 + \ln(6)] \\ \hat{y}(4.3) &= 7.737\ldots\end{aligned}$$
- This answers the QotD, but is it a good answer?
- A better approximation could be made by taking smaller steps. If $h = 0.1$ instead of $h = 0.3$, then instead of moving from $\hat{y}(4)$ to $\hat{y}(4.3)$, we would do:
$$\hat{y}(4) \to \hat{y}(4.1) \to \hat{y}(4.2) \to \hat{y}(4.3).$$
So that means:
$$\begin{aligned}\hat{y}(4.1) &= \hat{y}(4) + h \cdot f(4, \hat{y}(4))) \\ \hat{y}(4.2) &= \hat{y}(4.1) + h \cdot f(4.1, \hat{y}(4.1)) \\ \hat{y}(4.3) &= \hat{y}(4.2) + h \cdot f(4.2, \hat{y}(4.2))\end{aligned}$$
Plugging in the numbers starting at $(4, 6)$ with $h = 0.1$ gives:

y	Euler's Method
$\hat{y}(4)$	6
$\hat{y}(4.1)$	$6.057918 = 6 + 0.1[4 + \ln(6)]$
$\hat{y}(4.2)$	$6.116835 = 6.057918 + 0.1[4.1 + \ln(6.057918)]$
$\hat{y}(4.3)$	$6.176753 = 6.116835 + 0.1[4.2 + \ln(6.116835)]$

Taylor series point of view

- The general DE:
$$y' = f(t, y)$$
- The solution to the DE satisfies
$$y(t+h) = y(t) + y'(t)(t+h-t) + \frac{1}{2}y''(\alpha)h^2$$
for some $\alpha \in [t, t+h]$.
- Since $y'(t) = f(t, y)$,
$$y(t+h) = y(t) + f(t, y)h + O(h^2)$$
- So the error is
$$\hat{y}(t+h) - y(t+h) = \frac{1}{2}y''(\alpha)h^2$$
- That is the error created in one step.
- Total number of steps is $1/h$, so (roughly speaking) error is $O(h)$. Call Euler's method a *first order* method

34.2 Error in Euler's method

Continuous functions

$$\lim_{x \to a} f(x) = f(a)$$

- Continuity does not say how quickly $f(x)$ approaches $f(a)$
- Lipschitz continuity does

Definition 76 (Lipschitz continuity)
A function f is **Lipschitz continuous** there exists L such that for any x_1 and x_2 in the domain $|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$. Call L the Lipschitz constant.

Example

- Show that $f(x) = |2x|$ is Lipschitz constant with parameter 2.

  Proof: Let $x_1, x_2 \in \mathbb{R}$. Then

  $$|f(x_1) - f(x_2)| = |2x_1 - 2x_2| = 2|x_1 - x_2|,$$

  so we are done! $\square$.

- Note $|2x|$ not differentiable at 0.
- Not all Lipschitz functions are differentiable, all continuously differentiable functions over a compact domain are Lipschitz

Lemma 54
Suppse $f \in C^1$ and $|f'(x)| \leq L$ for all $x \in [a, b]$. Then f is Lipschitz with parameter L over $[a, b]$.

Lemma 55 (Euler's method error)
Given $y(t_0) = y_0$, and $y'(t) = f(t, y)$, let $\hat{y}(t)$ be the Euler's method estimate of $y(t)$. For fixed t, let $g_t(y) = f(t, y)$. Then suppose for all t, $y''(t) \leq M$, and for all t, g_t is Lipschitz continuous with parameter L. Then

$$|\hat{y}(t) - y(t)| \leq h\frac{M}{2L}\left(\exp(L(t - t_0) - 1\right).$$

Notes

- Error bound rarely used in practice
- The $(1/2)M$ comes from the $(1/2)y''$ term in the Taylor series expansion
- Error is linear in h
- Error grows exponentially in $t - t_0$: the farther away you are from the starting point, the worse the error can be
- In practice, this bound is pessimistic

34.3 Improving upon Euler's method

Backward Euler

- In Euler: $\hat{y}(t + h) = \hat{y}(t) + f(t, \hat{y}(t))h$
- Called "forward Euler" because f evaluated at $(t, \hat{y}(t))$
- Backward Euler: Solve $\hat{y}(t + h) = \hat{y}(t) + f(t, \hat{y}(t + h))h$ for $\hat{y}(t + h)$
- Mathematicians call backwards Euler an *implicit* method, while forwards Euler is an *explicit* method

Runge-Kutta

- Euler estimates $y(t+h)$ using times $\{t, t+h\}$
- Runge-Kutta uses times $\{t, t+h/n, t+2h/n, \ldots, t+nh/n\}$
- Gets higher order error
- Example ($n=2$)
$$\hat{y}(t+h) = \hat{y}(t) + hf(t+h/2, \hat{y}(t) + (1/2)hf(t, \hat{y}(t)))$$

Multistep

- Gets higher order convergence without using extra times
- Uses $\hat{y}(t), \hat{y}(t-h), \ldots, \hat{y}(t-nh)$ to find $\hat{y}(t+h)$
- Example: $n=2$ (Adams-Bashforth)
$$\hat{y}(t+h) = \hat{y}(t) + (3/2)hf(t, \hat{y}(t)) - (1/2)hf(t-h, \hat{y}(t-h))$$
- Procedure: Use Euler to get $\hat{y}(t_0+h)$, then use $y(t_0)$, $\hat{y}(t_0+h)$ from there with Adams-Bashforth
- Generally best way to do estimation

Method	$\hat{y}(1)$
Euler $h=0.1$	0.39844
Euler $h=0.01$	0.4256
Adams-Bashforth $h=0.1$	0.4275
Adams-Bashforth $h=0.01$	0.428571
Adams-Bashforth $h=0.005$	0.428578

- To 4 sig figs, $\boxed{0.4285}$

35 Polar Coordinates

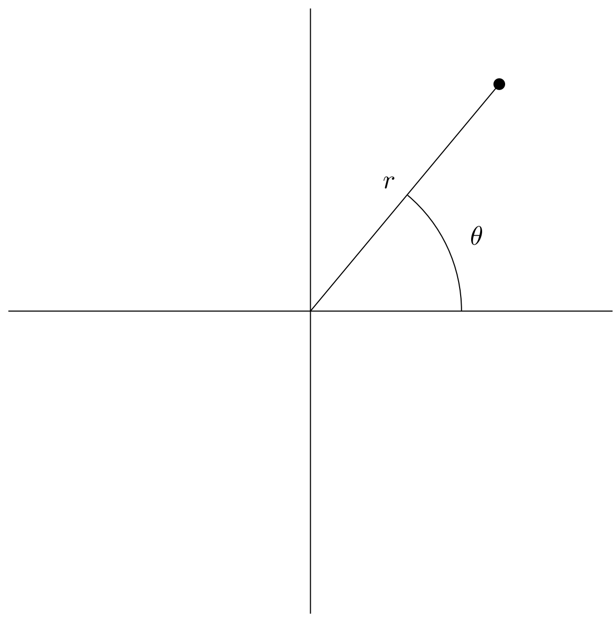
Question of the Day Find the area enclosed by a four leaf rose, $r = \cos(2\theta)$.

Today

- Polar coordinates
- Areas using polar coordinates

Polar coordinates

- Rectangular (x, y)
- Polar coordinates (r, θ)



- Conversion between polar and rectangular:

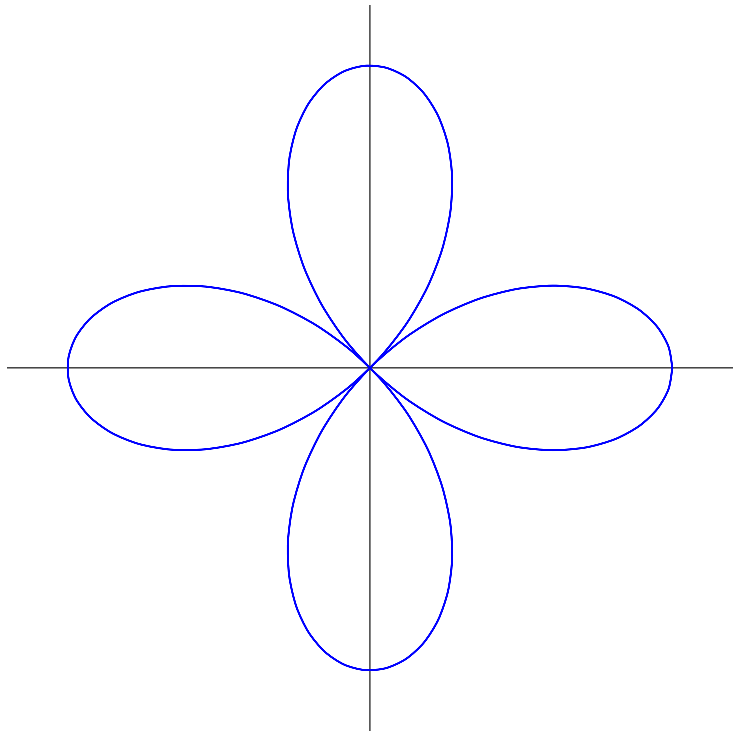
$$x = r\cos(\theta) \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin(\theta) \qquad \theta = \arctan(y/x)$$

To graph $r = \cos(2\theta)$

- Two common approaches
  1: Convert equation to rectangular coordinates
  2: Or just plug in various values of θ to find r and plot.

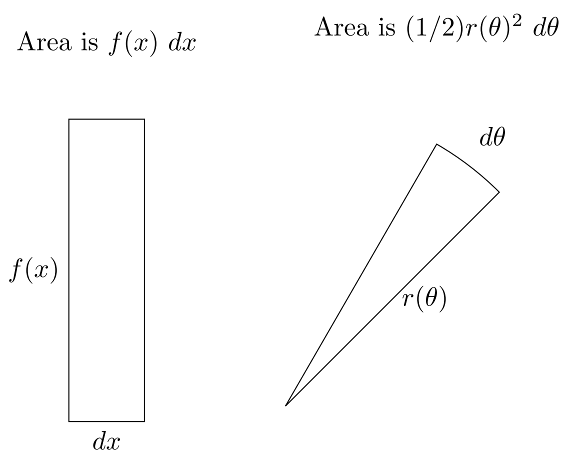
θ	r
0	1
$\pi/8$	$\cos(\pi/4) = \sqrt{2}/2$
$\pi/4$	$\cos(\pi/2) = 0$
$3\pi/8$	$\cos(3\pi/4) = -\sqrt{2}/2$
$\pi/2$	$\cos(\pi) = -1$
$\vdots$	$\vdots$
2π	$\cos(4\pi) = 1$

- Wolfram alpha: `plot r = cos(2*theta)`



Areas w/ polar coordinates

- Rectangular coordinates: differential element look like rectangles
- Polar coordinate: differential element looks like pizza slices



  - For rectangular coord area between $f(x)$ and x-axis is

$$\int_{x=a}^{b} |f(x)|\ dx.$$

Fact 21
In polar coordinates, area between the origin and the $r(\theta)$ is

$$\int_{\theta=a}^{b} \frac{1}{2} r(\theta)^2 \, d\theta.$$

Qotd

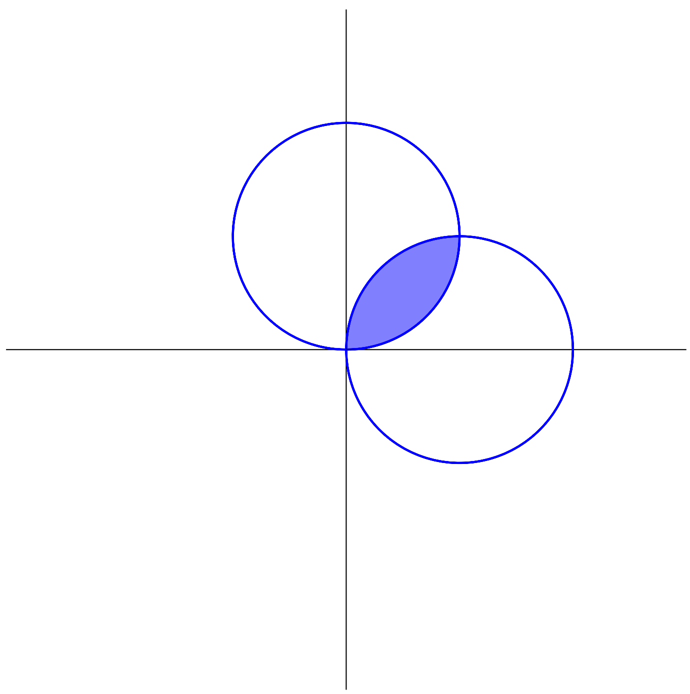
$$\begin{aligned} I &= \int_0^{2\pi} \frac{1}{2}\cos^2(2\theta) \, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos(4\theta) \, d\theta \\ &= \frac{1}{4}\left[\theta + \frac{1}{4}\sin(4\theta)\right] |_0^{2\pi} \\ &= \frac{2\pi}{4} = \frac{\pi}{2} \approx \boxed{1.570} \end{aligned}$$

Area between curves

- Remember integral sweeps out area from origin to curve
- Example: find the area common to circles

$$r = a\sin(\theta) \text{ and } r = a\cos(\theta),$$

where a is a constant.



- First, why are these circles?

$$x = r\cos(\theta) \text{ so } x = a\cos(\theta)\cos(\theta) \text{ so } x = a\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)$$

$$y = r\sin(\theta) \text{ so } y = a\cos(\theta)\sin(\theta) \text{ so } y = a\left(\frac{1}{2}\sin(2\theta)\right)$$

- Which means as θ runs from 0 to π, we trace out a circle centered at $((1/2)a, 0)$.
- Similarly, $r = \sin(2\theta)$ traces out a circle centered at $(0, (1/2)a)$
- Looking at our region, we really have two integrals!

$$\int_0^{\pi/4} \frac{1}{2}a^2 \sin^2(\theta)\ d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}a^2 \cos^2(\theta)\ d\theta.$$

The result is:

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} \frac{1}{2}a^2 \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)\ d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}a^2 \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\ d\theta \\
&= \frac{1}{2}a^2 \left[\left(\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{\pi/4} + \left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Bigg|_{\pi/4}^{\pi/2}\right] \\
&= \boxed{\frac{1}{8}a^2(\pi - 2).}
\end{aligned}$$

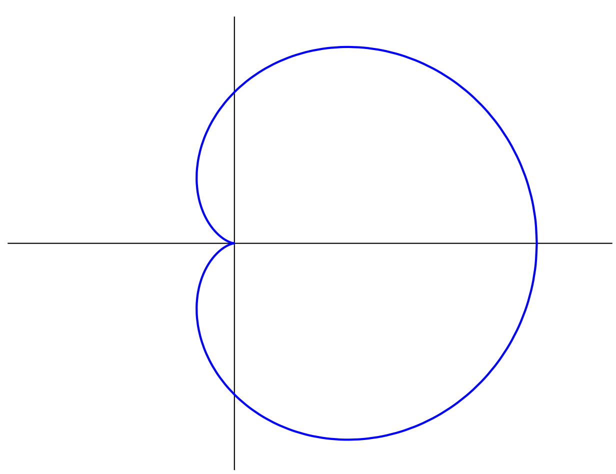
Remember for polar area:

- Scan radially outward from origin
- Start at $\theta = 0$ (east), as θ increases move counterclockwise.

Example Find the area of the top half of a cardioid

$$r = 1 + \cos(\theta)$$

[A picture can help in finding the limits of θ.]



- Since $\theta = 0$ due east, and $\theta = \pi$ due west, $\theta \in [0, \pi]$.

$$
\begin{aligned}
\text{Area} &= \int_0^{\pi} \frac{1}{2}\left(1+\cos(\theta)\right)^2 \; d\theta \\
&= \frac{1}{2}\int_0^{\pi} 1 + 2\cos(\theta) + \cos^2(\theta) \; d\theta \\
&= \frac{1}{2}\int_0^{\pi} 1 + 2\cos(\theta) + \frac{1}{2} + \frac{1}{2}\cos(2\theta) \; d\theta \\
&= \frac{1}{2}\left(\theta + 2\sin(\theta) + \frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{\pi} \\
&= \boxed{\frac{3\pi}{4} \approx 2.356}.
\end{aligned}
$$

36 Complex Numbers

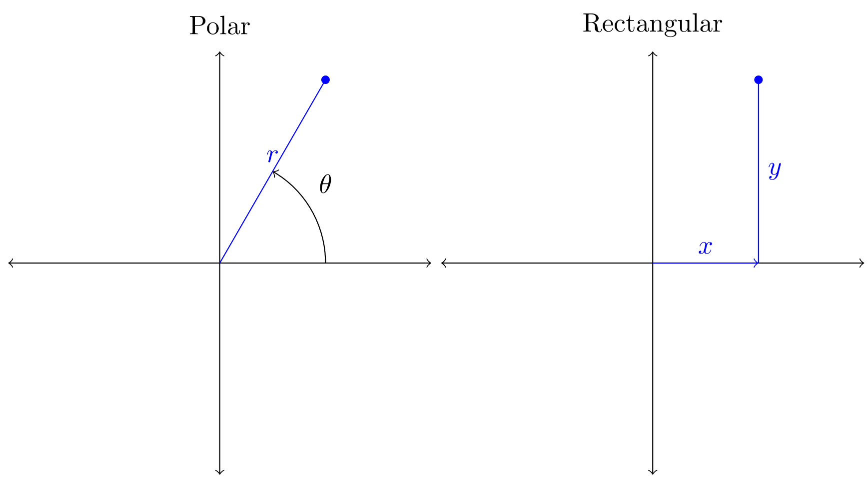
Question of the Day What is $e^{i\pi/4}$?

Today

- Complex Numbers

QotD

- So far, exponential defined for real numbers
- Real #'s only exist in one dimension
- Complex numbers exist in two dimensions
- Two common ways to describe points in a plane: Polar, and Rectangular



Polar multiplication

- Let z_1, z_2 be two complex #'s
- Let z_1 be r_1 distance from origin, CCW angle θ_1
- Let z_2 be r_2 distance from origin, CCW angle θ_2
- Then $z_1 \cdot z_2$ is $r_1 \cdot r_2$ from origin, angle $\theta_1 + \theta_2$

Example

Polar and rectangular

- Polar rep: z has distance r and angle θ
- Rectangular rep: $z = x + iy$

Raising i to powers

- $i^1 = i$ (turn 90 degrees CCW once)
- $i^2 = -1$ (turn 90 degrees CCW twice)
- $i^3 = -i$ (turn 90 degrees CCW three times)
- $i^4 = 1$ (turn 90 degrees CCW four times)

Taylor series

- Recall three Taylor series:

$$\begin{aligned}\exp(w) &= 1 + w + w^2/2! + w^3/3! + \cdots \\ \sin(w) &= w - w^3/3! + w^5/4! + \cdots \\ \cos(w) &= 1 - w^2/2! + w^4/4! - \cdots\end{aligned}$$

- So let's just plug in $w = i\pi/4$:

$$\begin{aligned}\exp(i\pi/4) &= 1 + i\pi/4 + i^2(\pi/4)^2/2! + i^3(\pi/4)^3/3! + i^4(\pi/4)^4/4! + \cdots \\ &= 1 - (\pi/4)^2/2! + (\pi/4)^4/4! - \cdots \\ &\quad + i\left[\pi/4 - (\pi/4)^3/3! + (\pi/4)^5/5! - \cdots\right] \\ &= \cos(\pi/4) + i\sin(\pi/4).\end{aligned}$$

- This gives us a way to convert back and forth between rectangular complex numbers and polar complex numbers!

Fact 22
For any θ:

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta).$$

Definition 77
Complex numbers of the form $z = x + i \cdot 0$ are called real, while $z = 0 + iy$ are called imaginary.

Note: Usual rules for exponents apply:

$$\exp(z_1)\exp(z_2) = \exp(z_1 + z_2),\ \exp(z_1)^{z_2} = \exp(z_1 \cdot z_2).$$

Example: What are the solutions to $x^3 = 1$?

Real solution: $x = 1$.
Want $[re^{i\theta}]^3 = 1 = 1 \cdot e^0$.
Gives two equations (note r is always real):

$$\begin{aligned}r^3 &= 1 & r &= 1 \\ 3\theta &= 0.\end{aligned}$$

Normally, $3\theta = 0$ only has one solution, $\theta = 0$.
But remember θ is an angle!
So $0 = 2\pi = 4\pi = 6\pi = 8\pi = \cdots$

$$\begin{aligned}3\theta = 0 &\Rightarrow \theta = 0 \\ 3\theta = 2\pi &\Rightarrow \theta = 2\pi/3 \\ 3\theta = 4\pi &\Rightarrow \theta = 4\pi/3 \\ 3\theta = 6\pi &\Rightarrow \theta = 2\pi = 0 \\ 3\theta = 8\pi &\Rightarrow \theta = 8\pi/3 = 2\pi/3\end{aligned}$$

So you get three solutions,

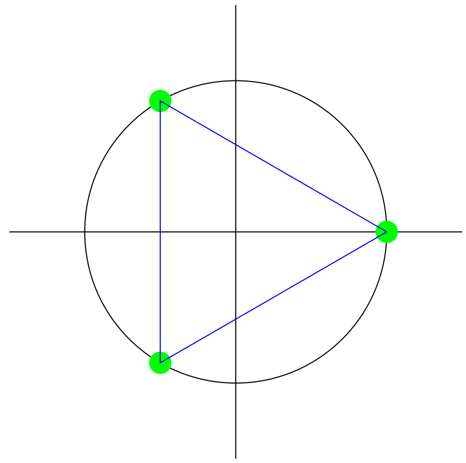
$$\theta \in \{0, 2\pi/3, 4\pi/3\}.$$

Hence the three solutions to $x^3 = 1$ are:

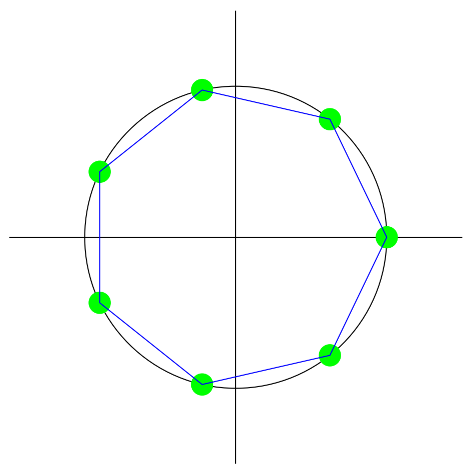
$$\boxed{1 = \exp(0), \exp(i \cdot 2\pi/3), \exp(i \cdot 4\pi/3).}$$

Or you could use trigonometry to say:

$$\boxed{1, -1/2 + i\sqrt{3}/2, -1/2 - i\sqrt{3}/2}.$$



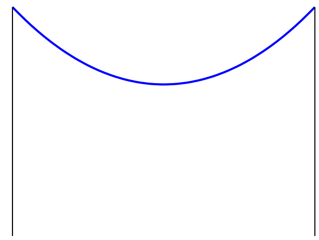
Solutions to $x^7 = 1$ form a regular septagon:



Fact 23 (Fundamental Theorem of Algebra)
A degree n polynomial $P(x)$ always has at least one complex solution.

37 Hyperbolic Functions

Question of the Day What is the shape of a wire hung between 2 poles?

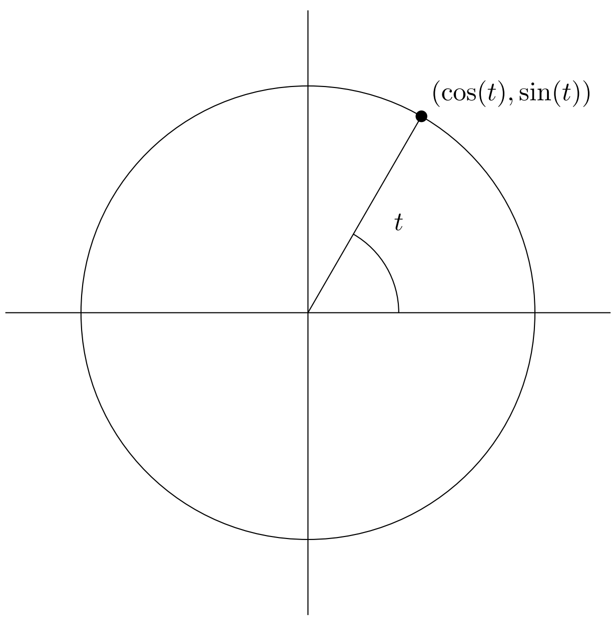


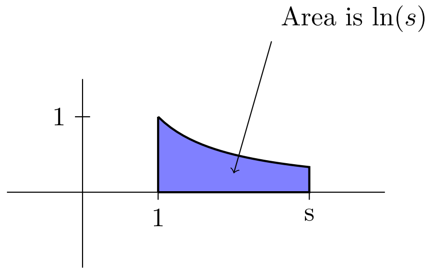
Today

- Hyperbolic functions

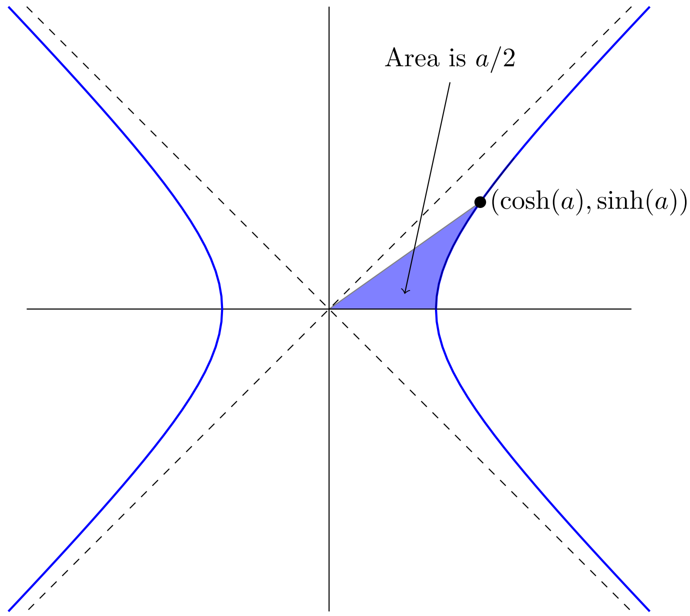
Hanging wire or chain

- Shape not quite a parabola
- Called a **catenary**
- It is an example of a hyperbolic function
- Recall that several functions are defined geometrically





- Hyperbolic functions are a combination of these approaches



- Notation

$$\begin{aligned} \cosh(a) &= \text{hyperbolic cosine} && \text{(pronounced kosh of } a) \\ \sinh(a) &= \text{hyperbolic sine} && \text{(pronounced cinch of } a) \end{aligned}$$

- Pythagorean identities change a bit:

$$\cos^2(a) + \sin^2(a) = 1$$
$$\cos^2(a) - \sin^2(a) = 1$$

- Can also be defined algebraically.

Definition 78
The **hyperbolic cosine** and **hyperbolic sine** functions are

$$\cosh(a) = \frac{1}{2}\left(e^a + e^{-a}\right).$$
$$\sinh(a) = \frac{1}{2}\left(e^a - e^{-a}\right).$$

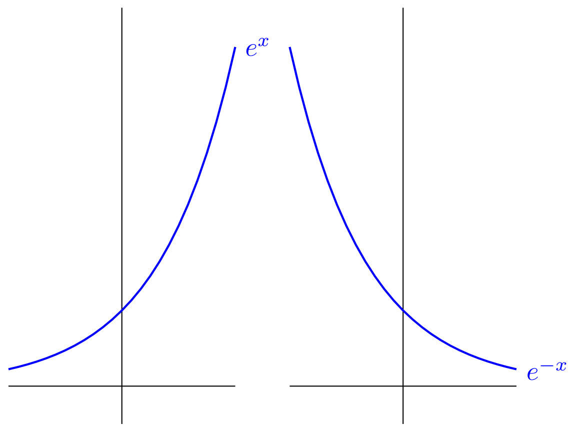
Fact 24
The derivative of sinh is cosh. The derivative of cosh is sinh. Hence $\mathrm{ad}_x(\sinh(x)) = \cosh(x)$ and $\mathrm{ad}_x(\cosh(x)) = \sinh(x)$.

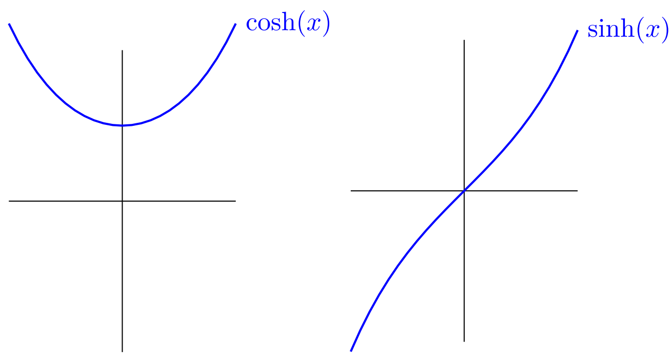
Proof.

$$\frac{d}{da}\sinh(a) = \frac{1}{2}(e^a - (-e^{-a})) = \cosh(a)$$
$$\frac{d}{da}\cosh(a) = \frac{1}{2}(e^a + (-e^{-a})) = \sinh(a).$$

□

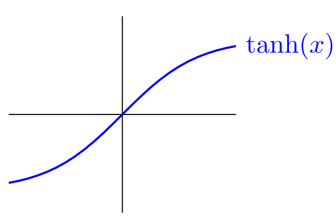
The graphs:





Definition 79
The **hyperbolic tangent** (written $\tanh(a)$) is

$$\tanh(a) = \frac{\sinh(a)}{\cosh(a)} = \frac{e^a - e^{-a}}{e^a + e^{-a}}.$$



Hyperbolic for antidifferentiation

Fact 25
Let $\sinh^{-1}(x)$ be the inverse hyperbolic sine function. Then

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}.$$

Proof. Let $y = \sinh^{-1}(x)$ so $x = \sinh(y)$. Then

$$\frac{dy}{dx} = \frac{1}{dx/dy} = 1/\cosh(y) = 1/\sqrt{1+\sinh^2(y)}.$$

This last is just $1/\sqrt{1+x^2}$. $\square$

Can write sinh in terms of exp, can write $\sinh^{-1}$ in terms of ln.

Fact 26
$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$.

Example

$$\begin{aligned}\int_3^4 \frac{1}{1+x^2}\,dx &= \sinh^{-1}(x)|_3^4 = \sinh^{-1}(4) - \sinh^{-1}(3)\\ &= \ln(4+\sqrt{17}) - \ln(3+\sqrt{10})\\ &\approx 0.2762\end{aligned}$$

Fact 27
For the inverse hyperbolic cosine:

$$\begin{aligned}\frac{d}{dx}\cosh^{-1}(x) &= \frac{1}{\sqrt{x^2-1}}\\ \cosh^{-1}(x) &= \ln(x+\sqrt{x^2-1}) \text{ when } x \geq 1.\end{aligned}$$

Back to the chain

Definition 80
A **catenary** is any function of the form

$$f(x) = k_1 \cosh(x/k_1) + k_2,$$

where k_1 and k_2 are constants.

Example 20 foot high poles 30 feet apart have a chain suspended between them. At the middle of the chain the height is 15 feet. What is the equation that describes the chain?

- Collate info: $f(15) = f(-15) = 20$, $f(0) = 15$.

$$\begin{aligned}f(0) = 15 &= k_1\cosh(0/k_1) + k_2\\ f(15) = 20 &= k_1\cosh(15/k_1) + k_2\end{aligned}$$

- Subtract second equation from first to get:

$$f(15) - f(0) = 5 = k_1[\cosh(15/k_1) - 1].$$

- This is an example of a *transcendental equation*, no analytic solution
- Numerical methods give:

$$k_1 = 23.28\ldots \text{ and } k_2 = -8.288\ldots.$$

38 Probability

Question of the Day Suppose X is a continuous random variable with density $2\exp(-2s)\mathbb{1}(s \geq 0)$. Find $\mathbb{P}(X \in [2,3])$.

Today

- Integrals for probabilities
- Integrals for normalizing constants
- Integrals for expectation and variance

Fact 28
For a continuous random variable X with density f_X, a and b in $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ with $a < b$, and continuous function g:

$$\begin{aligned}
\mathbb{P}(X = a) &= 0 \\
\mathbb{P}(-\infty < X < \infty) &= 1 \\
\mathbb{P}(a < X < b) &= \int_a^b f_X(s)\ ds \\
\mathbb{E}(g(X)) &= \int_{-\infty}^{\infty} g(s) f_X(s)\ ds.
\end{aligned}$$

38.1 Probabilities

Qotd

- In the qotd, since X is continuous, $\mathbb{P}(X = 5) = 0$
- $\mathbb{P}(X = -1) = 0$, $\mathbb{P}(X = 10) = 0$.
- However, for nontrivial intervals:

$$\begin{aligned}
\mathbb{P}(2 \leq X \leq 3) &= \mathbb{P}(2 < X < 3) \\
&= \int_2^3 f_X(s)\ ds \\
&= \int_2^3 2\exp(-2s)\ ds \\
&= \left. \frac{2\exp(-2s)}{-2} \right|_2^3 \\
&= \exp(-4) - \exp(-6) \approx \boxed{0.01583}.
\end{aligned}$$

- Sometimes $a = -\infty$ or $b = \infty$. So

$$\mathbb{P}(X > 4) = \mathbb{P}(4 < X < \infty) \text{ and } \mathbb{P}(X < 2) = \mathbb{P}(-\infty < X < 2),$$

so the probability rule leads to improper integrals in this case.

38.2 Expectation

Definition 81
The **mean**, **expected value**, **average**, or **expectation** of a function of a continuous random variable with density $f_X(s)$ are all:

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(s) \cdot f_X(s)\ ds.$$

- Inside the integral, replace every instance of X with the dummy variable, and then multiply by the density of X.
- For example, if $f_X(s) = 2\exp(-2s)\mathbb{1}(s \geq 0)$:

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} s \cdot 2\exp(-2s)\mathbb{1}(s \geq 0)\ ds \\
\mathbb{E}[X^2] &= \int_{\mathbb{R}} s^2 2\exp(-2s)\mathbb{1}(s \geq 0)\ ds \\
\mathbb{E}[e^X] &= \int_{\mathbb{R}} e^s 2\exp(-2s)\mathbb{1}(s \geq 0)\ ds.
\end{aligned}$$

- Let's tackle the first integral. First get rid of the indicator function:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} s \cdot 2\exp(-2s)\mathbb{1}(s \geq 0)\ ds = 2\int_0^{\infty} s\exp(-2s)\ ds.$$

- Now use integration by parts with $f(s) = s$, $g'(s) = \exp(-2s)$, so $f'(s) = 1$ and $g(s) = \exp(-2s)/(-2)$:

$$\begin{aligned}
\mathbb{E}[X] &= 2\left[s\exp(-2s)/(-2)|_0^{\infty} - \int_1^{\infty} \exp(-2s)/(-2)\ ds \right] \\
&= 2\left[\lim_{b \to \infty} b\exp(-2b)/(-2) - 0 + \int_1^{\infty} \exp(-2s)/2\ ds \right] \\
&= \int_1^{\infty} \exp(-2s)\ ds \\
&= \left.\frac{\exp(-2s)}{-2}\right|_1^{\infty} \\
&= \lim_{b \to \infty} \frac{\exp(-2b)}{-2} - \frac{e^0}{-2} \\
&= 1/2 = \boxed{0.5000}
\end{aligned}$$

- To solve $\mathbb{E}(X^2)$ would need to use integration by parts twice.
- To solve $\mathbb{E}(e^X)$, note that

$$\mathbb{E}(e^X) = \int_{\mathbb{R}} e^s \cdot 2e^{-2s}\ ds = \int_{\mathbb{R}} 2e^{-s}\ ds.$$

38.3 Normalizing densities

Definition 82
A function $f \geq 0$ where

$$\int_{-\infty}^{\infty} f(s)\ ds = 1$$

is called a *normalized density*.
Definition 83
A function $g \geq 0$ where

$$\int_{-\infty}^{\infty} g(s)\ ds < \infty$$

is called a *unnormalized density*. Call $\int_{\mathbb{R}} g(s)\ ds$ the *normalizing constant*.

Fact 29
If g is an unnormalized density, then $g/\int_{\mathbb{R}} g(s)\ ds$ is a normalized density.

Ex: Find the normalized density of $g(s) = (1/s^5)\mathbb{1}(s \geq 1)$.

Answer: Find the normalizing constant:

$$\begin{aligned}
\int_{-\infty}^{\infty} g(s)\ ds &= \int_{-\infty}^{\infty} \frac{1}{s^5}\mathbb{1}(s \geq 1)\ ds \\
&= \int_{1}^{\infty} \frac{1}{s^5}\ ds \\
&= \left.\frac{s^{-4}}{-4}\right|_{1}^{\infty} \\
&= \lim_{b\to\infty} b^{-4}/(-4) - 1^{-4}/(-4) \\
&= 1/4.
\end{aligned}$$

Hence

$$\frac{s^{-5}\mathbb{1}(s \geq 1)}{1/4} = \boxed{4s^{-5}\mathbb{1}(s \geq 1)}$$

is the normalized density.